

# Emergence of Space-Time from Topologically Homogeneous Causal Networks

Giacomo Mauro D'Ariano and Alessandro Tosini

*QUIT Group, Dipartimento di Fisica "A. Volta", via Bassi 6, I-27100 Pavia,  
Italy and Istituto Nazionale di Fisica Teorica e Nucleare, Sezione di Pavia.*

In this paper we study the emergence of Minkowski space-time from a causal network. Differently from previous approaches, we require the network to be topologically homogeneous, so that the metric is derived from pure event-counting. Emergence from events has an operational motivation in requiring that every physical quantity—including space-time—be defined through precise measurement procedures. Topological homogeneity is a requirement for having space-time metric emergent from the pure topology of causal connections, whereas physically corresponds to the universality of the physical law. We analyze in detail the case of  $1 + 1$  dimension. Coordinate systems are established via an Einsteinian protocol, and lead to a digital version of the Lorentz transformations. In a computational analogy, the foliation construction can also be regarded as the synchronization with a global clock of the calls to independent subroutines (corresponding to the causally independent events) in a parallel distributed computation, and the Lorentz time-dilation emerges as an increased density of leaves within a single *tic-tac* of a clock, whereas space-contraction results from the corresponding decrease of density of events per leaf. The operational procedure of building up the coordinate system introduces an in-principle indistinguishability between neighboring events, resulting in a network that is coarse-grained, the thickness of the event being a function of the observer clock. The present simple cinematocal construction does not extend straightforwardly to space dimension greater than one, due to anisotropy of the maximal speed: this issue is cured by a superposition of causal paths, specializing the causal network to a quantum computational one.

## I. INTRODUCTION

Our everyday way of looking at space-time as a stage for physical events conflicts with the requirement of defining all physical quantities—including space and time—through precise measurement protocols. This means that we should more properly regard space-time as emerging from events, instead of pre-existing them. The operational definition of space-time is defined by the protocol that sets up the coordinate system. For example, in the Einstein protocol light pulses are sent back and forth between different locations: at the place where the signal has been originated, from the arrival time of the reflected signal one infers both the distance and the time of the remote event of signal-reflection. The protocol shows how space-time is indeed a coherent organization of inferences based on a causal structure for events. The clock itself is just a sequence of events—a light pulse bouncing between two mirrors. The closest are the mirrors, the more precise is the clock, and the more refined is the coordinate system.

The above reasoning shows that ultimately space and time are defined through pure event-counting, precisely counting of *tic-tacs* of the observer's clock, and we are thus lead to regard space and time as emergent from the topology of the causal network of events. The events of the network should not be regarded as actual, but more generally as potential, and the fabric of space-time is precisely the network of causal links between them.

The idea of deriving space-time from purely causal structures is not new. Raphael Sorkin started an independent research line in quantum gravity based on this idea more than two decades ago [1]. This was motivated by the potentialities of the approach residing in

the natural discreteness of the causal network, which also provides a history-space for a path integral formulation [2, 3]. The possibility of recovering the main features of the space-time manifold—topology, differentiable structure, and conformal metric—has been shown, starting from discrete sets of points endowed with a causal partial ordering [4]. Along these lines, in an operational context Lucien Hardy has also formulated a *causaloid* approach [5], which considers the possibility of a dynamical treatment of the causal links.

In the causal-set approach of Sorkin events are randomly scattered in order to avoid occurrence of their sparseness in the boosted frames that would lead to violation of Lorentz invariance. Unfortunately such randomness assumption contradicts the same idea of space-time emergence, since conceptually a random scattering of events needs a prior notion of their position. On the contrary, a genuine space-time emergence would require the metric to originate from the pure topological structure of the causal network. The need of recovering the metric from pure event-counting rules out randomness even at the purely topological level. The homogeneity of topology has also a physical motivation, since it corresponds to the universality of the physical law. It will clearly lead to violation of Lorentz covariance on a scale corresponding to the minimal space and time distances between events: the idea, however, is to have covariance restored in the limit of very large number of events.

In this paper we analyze the mechanism of emergence of space-time from homogeneous topology in  $1 + 1$  dimensions. We derive a digital version of the Lorentz transformations along with the corresponding digital-analog conversion rule. A uniform foliation (boost) is achieved via a uniform local rule for synchronising causally independent

events. This corresponds to an Einstein clock with the *tac* and the *tac* having different relative lengths in number of events. The procedure is analogous to the synchronisation with a global clock of the calls to input-output disconnected subroutines in a parallel distributed computation, as noticed by Leslie Lamport [6]. The boost are determined by the relative lengths of the *tac* and *tac* of the clock, and the Lorentz time-dilation emergence corresponds to an increased number of leaves within a clock tic-tac, whereas space-contraction results from the corresponding decreased density of events per leaf, as shown in Ref. [7].

In this paper we will show how the digital coordinate system is obtained by the analogous of the Einstein's procedure, with signals sent back-and-forth to events from an observer clock. The procedure introduces an in-principle indistinguishability between neighbouring events, due to the limited precision of the clock, resulting in a network that is coarse-grained, with the event thickness also depending on the boost. The digital version of the Lorentz transformation is an integer relation which differs from the usual analog transformation by a multiplicative real constant corresponding to the event thickness. The composition rule for velocities is independent on such constant, and is the same in the analog and digital versions. Preliminary results of the present work has already been given in Ref. [8].

The above purely cinematocal construction does not extend straightforwardly to space dimension greater than one, mostly due to the anisotropy of the maximal speed arising in the analog frame [9]. Symmetry is restored in the limit of very large number of events when the network is quantum—e.g. a quantum cellular automata—in the limit of widely delocalized state-preparations. Thus, the quantum nature of the network seems to play a crucial role in reestablishing Lorentz covariance.

## II. SETTING UP THE DIGITAL COORDINATE SYSTEM

The first problem to address is which lattice should be adopted for the causal network. In our convention the causal arrow is directed from the bottom to the top of the network. The dimension of the emerging space-time corresponds to the graph-dimension of the network, which is the dimension of the embedding manifold such that all links can be taken as segments of straight line with the same length. We will require that the lattice be pure topology (namely with all events equivalent), corresponding to a locally homogeneous space-time, and with no redundant links. It is then easy to see that in 1+1 dimensions there are only three possible lattices: the square, the triangular, and the honeycomb lattices. The honeycomb lattice has two inequivalent types of events (having one input and two output links and viceversa), and the corresponding undressed topology—with each couple of connected inequivalent events merged into a

single event—reduces to the square lattice. The triangular lattice, on the other hand, has redundant causal links (the middle vertical ones). We are thus left with the square lattice.

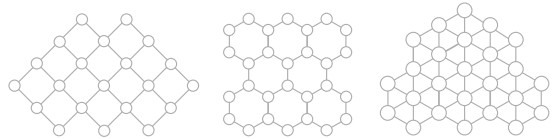


FIG. 1: The three possible homogeneous topologies for causal networks in 1+1 dimensions.

In the square-lattice network there are two types of link: toward left and toward right—shortly *l* and *r*. Two events are in the same position (for some boosted reference frame) if they are connected by a path made with a sequence of *l*-links followed by a sequence of *r*-links. (Notice that we always implicitly assume the link as oriented according to the causal arrow). When the two sequences contain the same number of links the reference is at rest.

A clock is a sequence of causally connected events periodically oscillating between two positions. For an Einstein clock the oscillation (*tac-tac*) is exactly the same couple of sequences of *l*- and *r*-links identifying events in the same position. The precision of the clock, namely the minimum amount of time that it can measure, is the number of links of a complete *tac-tac*. The *tac-tac* is indivisible, namely the sole *tac* (or *tac*) is not a complete measured time interval, since it involves two different positions.

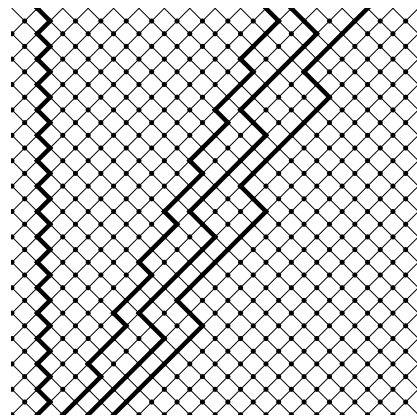


FIG. 2: Different clocks on the causal network. From the left to the right: the rest-frame clock, clock for a frame moving at digital speed  $v = \frac{1}{3}$ ,  $v = \frac{1}{3}$  with doubled imprecision,  $v = \frac{2}{7}$ .

In the following we will call *light signals* those sequences of events that are connected only by *l*-links or by *r*-links, namely making segments at 45 degrees with the horizontal in the network. Their “speed” is equal to “one event-per-step”, and is the maximum speed allowed by the causality of the network, since connecting events along a line making an angle smaller than 45° with the horizontal would require to follow some causal

connections in the backward direction from the output to the input. In this way in a homogeneous causal network causality suffices to guarantee a bound for the speed of information flow.

A reference frame is build up using the Einstein protocol as follows. From the start of the clock *tic-tac* a light signal is sent to an event in a different position and then received back at the clock. The intermediate time between the sending and the receiving event is taken as synchronous with the event at the turning point, and the number of *tic-tacs* divided by two is taken as the distance from the turning point and the clock conventionally located at the beginning of the *tic-tac* (see Fig. 3).

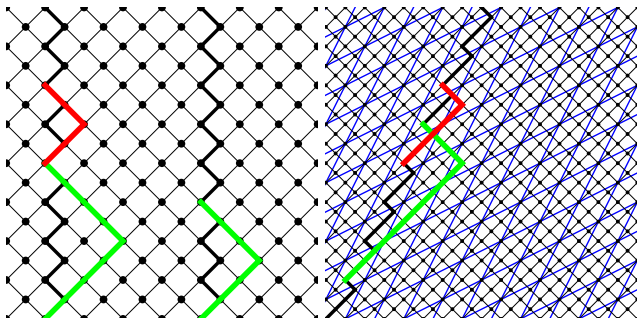


FIG. 3: Illustration of the Einstein protocol for establishing a reference frame. Left figure: rest-frame: in the clock on the right synchronisation occurs at the middle of the *tic-tac*, and the event is not discriminable from the neighbouring events. Right figure: boosted reference frame. The blue frame is the established reference frame.

Due to indivisibility of the *tic-tac*, we see that there are indiscernible events, for which the synchronisation occurs in the middle of the *tic-tac*. We are thus led to identify events merging them into coarse-grained events. This is done as follows. We identify the events along the *tic* and those along the *tac* so that the *tic-tac* is always regarded as the bouncing between two next neighbour events. Then we merge events into minimal sets so that the topology is left the invariant (see Figures 4 and 5). Notice that one has coarse-graining even in the rest frame, simply due to discreteness of time (see Figure 4).

### III. THE DIGITAL LORENTZ TRANSFORMATIONS

In the following we will take the clock *tic-tac* made with  $\alpha$  *l*-links followed by  $\beta$  *r*-links. As already said, the starting and ending events are considered at the same position, whence the corresponding reference is boosted with a velocity in the rest-frame corresponding to the ratio of the distances in space and time between the two ending points, namely

$$v = \frac{\alpha - \beta}{\alpha + \beta}. \quad (1)$$

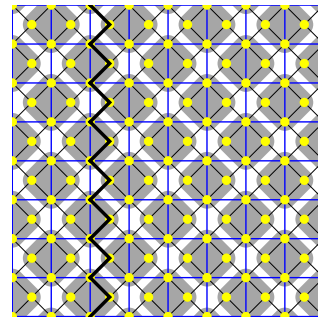


FIG. 4: Coarse-graining in the rest-frame. The coarse-graining due to boosts is trivial (depicted in yellow, containing just one event), whereas the coarse-graining associated to the intrinsic imprecision of the clock is not trivial (depicted in gray, containing four events).

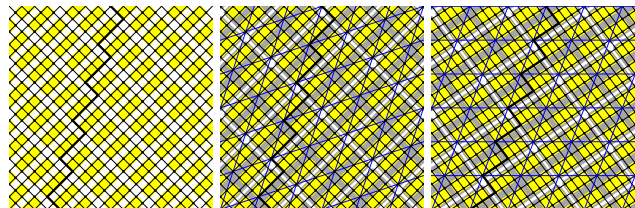


FIG. 5: Illustration of the coarse-graining procedure due to boost (in yellow) and to (intrinsic) imprecision of the clock. Left figure: events along the *tic* and events along the *tac* are identified in the boosted frame. Then events are merged into minimal sets so that the topology is left invariant (the merged events are again events of a square-lattice network). Center figure: the coarse graining associated to the intrinsic imprecision is added (in gray). Right figure: the circuit is stretched so to have all synchronous events on horizontal lines.

Thanks to invariance of topology under the boost coarse-graining, the above identity holds also for the motion relative to any boosted frame, whence, upon defining with  $\alpha_{12} = \alpha_2/\alpha_1 \in \mathbb{Z}^+$  and  $\beta_{12} = \beta_2/\beta_1 \in \mathbb{Z}^+$  for frames 1 and 2, and by  $v_{12} \in \mathbb{Z}^+$  the relative velocity of frame 2 with respect to frame 1, one has

$$v_{12} = \frac{\alpha_{12} - \beta_{12}}{\alpha_{12} + \beta_{12}}. \quad (2)$$

Now, by using the identities  $\alpha_{13} = \alpha_{12}\alpha_{23}$  and  $\beta_{13} = \beta_{12}\beta_{23}$  one has

$$v_{13} = \frac{\alpha^{12}\alpha^{23} - \beta^{12}\beta^{23}}{\alpha^{12}\alpha^{23} + \beta^{12}\beta^{23}}, \quad (3)$$

which by simple algebraic manipulations gives

$$v_{13} = \frac{\left(\frac{\alpha^{12}-\beta^{12}}{\alpha^{12}+\beta^{12}}\right) + \left(\frac{\alpha^{23}-\beta^{23}}{\alpha^{23}+\beta^{23}}\right)}{1 + \left(\frac{\alpha^{12}-\beta^{12}}{\alpha^{12}+\beta^{12}}\right)\left(\frac{\alpha^{23}-\beta^{23}}{\alpha^{23}+\beta^{23}}\right)} = \frac{v_{12} + v_{23}}{1 + v_{12}v_{23}}. \quad (4)$$

The last identity is the composition rule of parallel velocities (the only possibility in 1 + 1 dimension) in special relativity.

We use the Einstein protocol to construct the boosted coordinated system with respect to the rest-frame along with the relative coordinate systems between any couple of boosted frames. From a simple inspection of Figures 3 it is easy to see that the coordinates of an event transform from the frame 1 to the frame 2 as follows

$$\begin{aligned} t_1 &= \frac{1}{2}(\alpha_{12} + \beta_{12})(t_2 + v_{12}s_2), \\ s_1 &= \frac{1}{2}(\alpha_{12} + \beta_{12})(s_2 + v_{12}t_2). \end{aligned} \quad (5)$$

Eq. (5) is easy to check with frame 1 as the rest frame, and invariance of topology with boost guarantees that it also holds between any couple of boosted frames.

Upon defining the following constant depending on the clocks of the two frames

$$\chi_{12} := \sqrt{\alpha_{12}\beta_{12}}, \quad (6)$$

and using the identity

$$\frac{1}{2}(\alpha_{12} + \beta_{12}) = \frac{\chi_{12}}{\sqrt{1 - v_{12}^2}}, \quad (7)$$

we obtain the *digital* Lorentz transformations

$$t_1 = \chi_{12} \frac{t_2 + v_{12}s_2}{\sqrt{1 - v_{12}^2}}, \quad s_1 = \chi_{12} \frac{s_2 + v_{12}t_2}{\sqrt{1 - v_{12}^2}}. \quad (8)$$

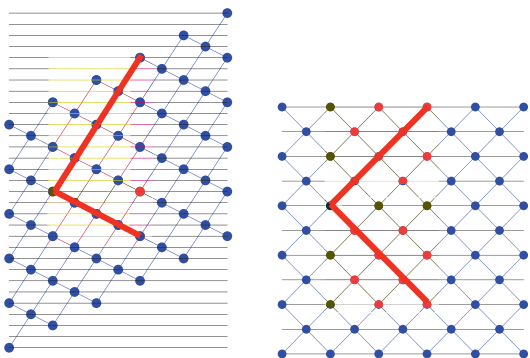


FIG. 6: The mechanism for the digital Lorentz time-dilation and space-contraction given in Ref. [7], here for a boost with  $v = \frac{1}{2}$ , corresponding to a digital time-dilation by a factor 2 (analog factor  $2/\sqrt{3}$ ) and space-contraction by a factor  $1/2$  (compare with the same factors in Eqs. (8)).

Eq. (8) differ from the usual *analog* Lorentz transformations by the multiplicative factor  $\chi_{12}$ , which is logically required to make the transformations rational, compensating the irrationality of the boost factor  $\sqrt{1 - v_{12}^2}$ . The digital-analog conversion is thus just a rescaling of both space and time coordinates by the factor  $(\alpha\beta)^{\frac{1}{2}}$  depending on the boost, which is exactly the square-root of the volume of the coarse-grained event measured as the number of rest-frame events that it contains. Such event volume also affects the Lorentz space-contraction and time-dilation factor, which in the digital case is given

by  $\frac{1}{2}(\alpha_{12} + \beta_{12})$ , whereas in the analog case is rescaled by the ratio of event volumes, leading to  $\frac{1}{2\sqrt{\alpha_{12}\beta_{12}}}(\alpha_{12} + \beta_{12})$ . Thus, for example, for  $\alpha_{12} = 1$  and  $\beta_{12} = 3$  corresponding to  $v_{12} = 1/2$  the digital factor is 2 whereas the analog one is  $2/\sqrt{3}$ . The digital factor agrees with that of the Lorentz time-dilation and space-contraction mechanism of Ref. [7], given in terms of increased density of leaves and corresponding decreased density of events per leaf, as illustrated in Fig. 6.

#### IV. CONCLUSIONS

In this paper we analyzed the mechanism of emergence of space-time from homogeneous topology in 1+1 dimensions, deriving the digital version of the Lorentz transformations along with the corresponding digital-analog conversion rule. The homogeneity of topology is a necessary ingredient for a genuine emergence of space-time from events, and physically represents the universality of the physical law. It is worth mentioning that such law is stripped of the conventionality of space and time homogeneity (see e.g. Ref. [10]). We have build the digital coordinate system using the Einstein's protocol, with signals sent back-and-forth to events from an observer clock. We found that the procedure introduces an in-principle indistinguishability between neighbouring events, due to the limited precision of the clock, resulting in a network that is coarse-grained, with the event thickness also depending on the boost. The digital version of the Lorentz transformation is an integer relation which differs from the usual analog transformation by a multiplicative real constant corresponding to the event thickness.

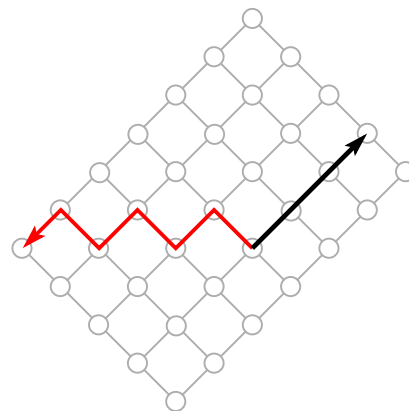


FIG. 7: BCC (1+2)-dimensional computational network: top view. Information must zig-zag to flow at the maximal speed in diagonal direction. This leads to a slow-down of a  $\sqrt{2}$  factor of the analog speed compared to cubic axis direction.

Our purely cinemtical construction used for one space dimension does not straightforwardly extend to larger dimensions. This is due to a general theorem of Tobias Fritz [9] showing that the polytope of points that can be reached in no more than  $N$  links in a periodic graph

does not approach a circle for large  $N$ . Since the polytope has necessarily distinguished directions, this means that there is no periodic graph for which this velocity set is isotropic. This result represents a no-go theorem for the emergence of an isotropic space from a discrete homogeneous causal network representing a *classical* information flow. For example, as shown in Fig. 7 in a causal network shaped as a BCC-lattice the fastest speed would be along the cubic axes, whereas along diagonals information should zig-zag, resulting in a slowdown by a factor  $\sqrt{2}$  or  $\sqrt{3}$ . Things, however, are different if one considers the possibility that information can flow in a superposition of paths, as in a homogeneous quantum computational network, i.e. a quantum cellular automata. A concrete example is given by the quantum cellular automata with BCC lattice of Bialynicki-Birula

[12], where the Dirac equation emerges in the limit of very de-localized states, thus restoring the full Lorentz covariance. An explicit local mechanism realizing the isotropy of information propagation will be given in a continuation of the present paper. Another possible way of regarding this solution of the dimensional conundrum is in terms of an anisotropic refraction index of vacuum of quantum nature [11], similar to the mass-dependent refraction index of vacuum in the 1+1 Dirac quantum cellular automata of Ref. [13], which is due to the unitarity of the evolution.

GMD acknowledges interesting discussions with Raphael Sorkin, Seth Lloyd, and Tobias Fritz. This work has been partially supported by PRIN 2008.

- 
- [1] L. Bombelli, J. H. Lee, D. Meyer, and R. Sorkin, *Phys. Rev. Lett* **59**, 521 (1987).
- [2] M. Fotini, gr-qc/0210086 (2002).
- [3] J. Henson, in *Approaches to Quantum Gravity: Towards a New Understanding of Space and Time*, Ed. D. Oriti (Cambridge University Press, Cambridge UK, 2006) (also gr-qc/0601121).
- [4] S. Surya, *Theor. Comp. Sc.* **405** 188 (2008).
- [5] L. Hardy, *J. Phys. A* **40** 3081 (2007).
- [6] L. Lamport, *Comm. ACM*, **21** 558 (1978).
- [7] G. M. D'Ariano, in CP1232 *Quantum Theory: Reconsideration of Foundations*, 5 ed. by A. Y. Khrennikov, (AIP, Melville, New York, 2010), pg. 3 (also arXiv: 1001.1088).
- [8] G. M. D'Ariano and A. Tosini, arXiv:1008.4805 (2010)
- [9] T. Fritz, *Velocity Polytopes of Periodic Graphs*, draft (2011).
- [10] D. Malament, *Noûs* **11** 293 (1977).
- [11] G. M. D'Ariano, *A Quantum-Digital Universe*, essay FQXi (2011).
- [12] I. Bialynicki-Birula, *Phys. Rev. D* **49** 6920 (1994).
- [13] G. M. D'Ariano, arXiv:1012.0756 (2010).