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State Discrimination with local Resources in the Fermionic Theory

Relatore:
PROF. PAOLO PERINOTTI

Correlatore:
DOTT. ALESSANDRO TOSINI

Tesi di laurea di:
MATTEO LUGLI
Matricola: 461121

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Alla mia famiglia

Contents

Chapter 1

Introduction

The present work investigates the problem of state discrimination by means of local operations and classical communication (LOCC) in the Fermionic theory. The Fermi-Dirac statistics describe half-integer spin particles whose applications, ranging from solid state to high energy physics, make of fermions one of the most fundamental notions of nature. However, the analysis of Fermionic systems as carriers of quantum information is rather recent and is still open to new investigations. The contemporary literature thoroughly tackles the Feynman problem of simulating fermions with qubits and the universality of Fermionic computation, providing sharp tools to better understand the theory. Thus, we review the Jordan-Wigner transformation, that is an isomorphism between the anticommuting Fermionic systems and the commuting qubits, along with the results concerning the entanglement between fermions. The LOCC protocols are firstly introduced in the quantum formalism as a subset of quantum instruments, which allow for local operations on systems with no shared entanglement resources, then translated into the Fermionic theory thanks to the Jordan-Wigner transformation. In the thesis, the Fermionic theory is treated as an instance of operational probabilistic theories (OPTs)[1].

An OPT is an operational language that expresses the possible connections between events, dressed with a probabilistic structure. In such a novel framework, we are interested in rigorously expressing the conditions for which two states are discriminable using some measurement protocols. It is known that quantum mechanics satisfies local discriminability, namely any state of a composite system can be probabilistically discriminated using only local measurements on the component systems. Moreover, orthogonal pure states can be perfectly (with probability one) discriminated using LOCC, whereas pairs of non-orthogonal pure states can be optimally discriminated by LOCC, i. e. with the optimal probability of discrimination. Finally, under certain hypothesis any pair of quantum states allows for optimal LOCC unambiguous discrimination, thus allowing for inconclusive outcomes. The validity of these results, which represent key features within quantum information and quantum computation, is still unexplored for the Fermionic theory.

In this thesis, we analyze the problem of state discrimination in the Fermionic theory by providing conditions under which states are distinguishable via LOCC.

[1] Cf. CDP10; DCP16.

We thereupon look into the prerequisites for Fermionic states in order to allow for optimal discrimination with local operations and classical communication, focusing on their operational interpretation. Furthermore, we show that optimal discrimination through LOCC is always possible provided some entanglement resources at disposal to the two parts. The long-term aim of this work is indeed to study the properties of the Fermionic theory in the context of wider studies still in progress on quantum and Fermionic cellular automata.

Chapter 2

State Discrimination and Local Operations in the Quantum Theory

We firstly introduce the quantum theory (QT) of information in the framework of OPTs. The fundamental axioms of the theory are defined according to von Neumann on finite-dimensional Hilbert spaces and illustrated along with the mathematical notation to describe quantum systems and operations. We then rigorously define the LOCC protocols, where many distant laboratories share a multipartite quantum system and are allowed to locally operate on their party but only classically communicate with each other. Finally, the literature results of perfect and optimal discrimination through LOCC are thoroughly reviewed.

We build up the theory starting from three postulates, as illustrated in the book [DCP16]:

1. To each system A we associate a complex Hilbert space $\mathcal{H}_A$. To the composition AB of systems A and B we associate the tensor product $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$.

2. To each state of system A corresponds a positive operator $\rho \in \mathrm{St}(\mathcal{H}_A)$ on $\mathcal{H}_A$ with $\mathrm{Tr}[\rho] \leq 1$.

3. Any map that satisfy all mathematical requirements for representing a transformation within the theory will actually be an admissible quantum transformations of the theory.

From all three assumptions we derive the well-know properties of the theory and the structure of state, effect and transformation sets, which we introduce hereafter.

Any state $\rho \in \mathrm{St}(\mathcal{H}_A)$, also known as a preparation of the system, is a positive operator $\rho \geq 0$ on the Hilbert space $\mathcal{H}_A$ such that $\mathrm{Tr}[\rho] \leq 1$. The set $\mathrm{St}(\mathcal{H}_A)$ boasts both a conic and convex structure. The states satisfying $\mathrm{Tr}[\rho] = 1$ are called *deterministic* and belongs to $\mathrm{St}_1(\mathcal{H}_A)$. We define a preparation test as a collection of preparations $\{\rho_i : \rho_i \in \mathrm{St}(\mathcal{H}_A)\}$ such that the *coarse graining*,

i. e.

$$\rho = \sum_i \rho_i, \tag{2.1}$$

is deterministic. Moreover, the conic structure emerges as soon as we consider a collection of sub-deterministic preparations $\{\rho_i : \mathrm{Tr}[\rho_i] < 1\}$ and conically combine them to attain

$$\sigma = \sum_i p_i \rho_i \quad \text{for} \quad p_i \geq 0, \tag{2.2}$$

where σ is a new preparation and must fulfill $\mathrm{Tr}[\sigma] \leq 1$. On the other hand, we define the *pure* states as those featuring rank equal to one, namely the projectors onto a one-dimensional subspace of $\mathcal{H}_A$. In the Dirac notation, we denote the pure preparations as $\rho = |\psi\rangle\langle\psi|$, for $|\psi\rangle \in \mathcal{H}_A$. Mixed states, on the contrary, have rank larger than one and are inherently linked to the convex combination of preparations

$$\tau = \sum_i p_i \rho_i \quad \text{for} \quad 0 \leq p_i \leq 1, \ \sum_i p_i = 1 \tag{2.3}$$

where ρ_i are some states and $\tau \in \mathrm{St}(\mathcal{H}_A)$.

Linear functionals on states are called effects and labeled as $E \in \mathrm{Eff}(\mathcal{H}_A)$. They are represented by positive operators dominated by the identity, that is $0 \leq E \leq I$; the identity, in turn, represents the deterministic effect. We denote the pairing between the state ρ and effect E on the same system through the Born rule as

$$p = (E|\rho) = \mathrm{Tr}[E\rho]. \tag{2.4}$$

Effects may be labeled as *atomic* when their rank is equal to one. We define as positive-operator valued measure (POVM) or effect test the collection of effects $\{E_i : 0 \leq E_i \leq I\}$ such that the coarse graining is the deterministic effect, namely $I = \sum_i E_i$.

We conclude our introduction with the definition of the quantum transformation $\mathcal{T} \in \mathrm{Transf}(A \to B)$ as the linear, completely-positive and trace-non-increasing map $\mathcal{T} : \mathrm{St}(\mathcal{H}_A) \to \mathrm{St}(\mathcal{H}_B)$. Both states and effects may be seen as transformations from and to the trivial system, respectively. Transformations that are trace-preserving, i. e. $\mathrm{Tr}[\mathcal{T}(\rho)] = \mathrm{Tr}[\rho]$, are called deterministic transformations or *quantum channels* and belongs to $\mathrm{Transf}_1(A \to B)$, whereas those that are only trace-non-increasing are usually named quantum operations. Thanks to Kraus' theorem we describe any transformation $\mathcal{T}$ through the set of operators $\{K_i : \mathcal{H}_A \to \mathcal{H}_B | \sum_i K_i K_i^\dagger \leq I\}$, called Kraus operators, such that

$$\mathcal{T}(\rho) = \sum_i K_i \rho K_i^\dagger. \tag{2.5}$$

Those maps featuring only one Kraus operator are called atomic. A collection of transformations $\{\mathcal{T}_i\}$ for which $\mathcal{T} = \sum_i \mathcal{T}_i$ is trace preserving is called a *quantum instrument* or transformation test. Quantum theory features the relevant relation that any deterministic and atomic transformation $\mathcal{T} \in \mathrm{Transf}(A \to A)$ is reversible, i. e. its only Kraus operator belongs to the set of unitary matrices $\mathsf{U}(\mathcal{H}_A)$.

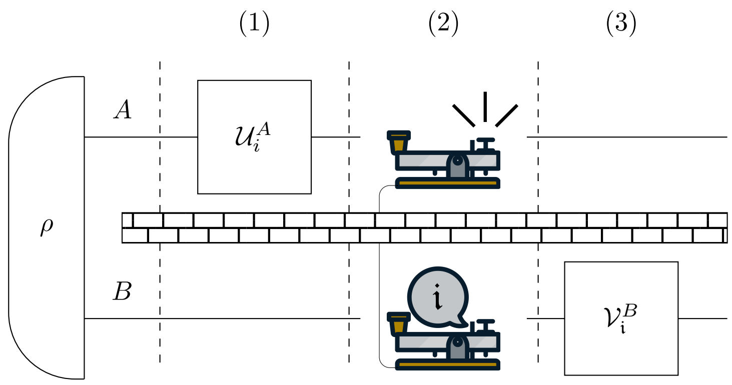

Figure 2.1: A schematic representation of a LOCC_1 protocol, between Alice and Bob sharing a bipartite system. The steps are labeled with numbers one to three: (1) Alice applies her instrument $\{\mathcal{U}_i\}$ on system A and reads $\mathtt{i}$, (2) Alice sends her outcome $\mathtt{i}$ to Bob through a classical channel and (3) Bob applies the corresponding deterministic transformation $\mathcal{V}_{\mathtt{i}}$. The telegram icon is made by Freepik from `www.flaticon.com`.

2.1 Local operations and classical communication

The paradigm of the "distant laboratory," where systems are shared among isolated parties, serves as typical setting for both theoretical and experimental studies on quantum entanglement. In QT, the axiom of local discriminability grants us the ability to distinguish probabilistically any pair of multipartite preparations just by means of local effects on the single parties, also known as separable effects (SEP). However, the set of separable effects does not provide a straightforward experimental interpretation and leads to some subtle issues, cf. [Ben+99]. If we allow the distant parties to locally apply transformations on their systems and classically communicate to each other in order to improve their measurement performances we come to the subset of local operations and classical communication (LOCC), which we introduce hereafter along the lines of the review [Chi+14]. Given a quantum instrument $\mathfrak{I} = \{\mathcal{T}_i\} \subset \mathrm{Transf}(Q \to Q')$ on a multipartite system $Q = ABC\ldots$ we define:

Definition 1 (One-way local)**.** The instrument $\mathfrak{I}^X$ is one-way local with respect to system X if each of its transformations has the form

$$\mathcal{T}_i = \mathcal{U}_i^X \otimes \left(\bigotimes_{S \neq X} \mathcal{V}_i^S \right), \tag{2.6}$$

where $\mathcal{U}_i^X \in \mathrm{Transf}(X \to X')$ and all $\mathcal{V}_i^S \in \mathrm{Transf}_1(S \to S')$ are deterministic.

The above setting has a plain operational interpretation: we firstly measure $\{\mathcal{U}_i^X\}$ on system X and classically communicate the output $\mathtt{i}$ to the other parties, who then apply the transformations $\mathcal{V}_{\mathtt{i}}^S$ $\forall S \neq X$ on the remaining systems. We define the one-round LOCC_1 set as all possible one-way local instruments

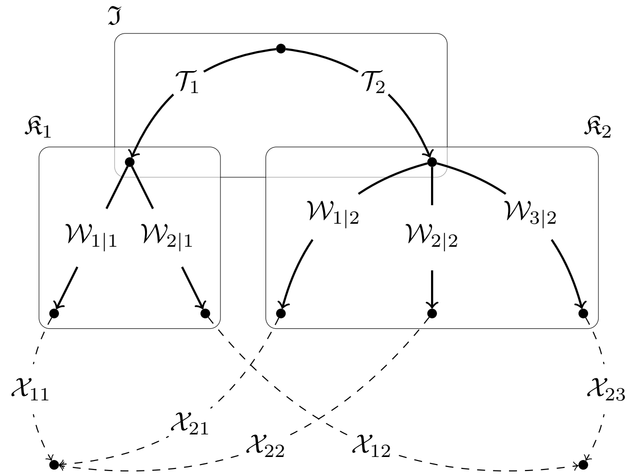

Figure 2.2: An illustration of a possible LOCC link between $\mathfrak{I} = \{\mathcal{T}_1, \mathcal{T}_2\}$ and $\mathfrak{T} = \{\mathcal{Z}_1, \mathcal{Z}_2\}$ is presented. We introduce the two instrument $\mathfrak{K}_1 = \{\mathcal{W}_{j|1}\}$ and $\mathfrak{K}_2 = \{\mathcal{W}_{j|2}$ for the two possible outcomes $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively. The resulting composite transformations $\mathcal{X}_{ij} = \mathcal{W}_{j|i} \circ \mathcal{T}_i$ are then coarse grained to assemble the operators $\mathcal{Z}_1 = \mathcal{X}_{11} + \mathcal{X}_{21} + \mathcal{X}_{22}$ and $\mathcal{Z}_2 = \mathcal{X}_{12} + \mathcal{X}_{23}$.

followed by a coarse-graining map, for which a pictorial representation is given in fig. 2.1. At this stage, we accordingly link several LOCC_1 instrument so that we define a protocol for multiple-round LOCC without loosing our earlier interpretation.

Definition 2 (LOCC link)**.** A quantum instrument $\mathfrak{T}$ is LOCC linked to $\mathfrak{I} = \{\mathcal{T}_i : i = 1 \dots n\}$ if there exists a collection of one-way local instruments $\mathfrak{K}_i = \{\mathcal{W}_{j|i} : j = 1 \dots m_i\}$ $\forall i$ for some system X_i such that $\mathfrak{T}$ is a coarse graining of the transformations

$$\mathcal{X}_{ij} = \mathcal{W}_{j|i} \circ \mathcal{T}_i. \tag{2.7}$$

The resulting protocol for $\mathfrak{T}$ is again straightforward. We apply the instrument $\mathfrak{I} = \{\mathcal{T}_i\}$ on the system and read the output i, we now choose the corresponding instrument $\mathfrak{K}_\mathrm{i}$ to measure among $\{\mathcal{W}_{j|\mathrm{i}}\}$ for $j = 1 \dots m$; an example is depicted in fig. 2.2.

Thanks to definition 2, we say that the instrument $\mathfrak{I}$ is a k-round LOCC_k if it is LOCC linked to some $\mathfrak{J} \in \mathsf{LOCC}_{k-1}$ for $k > 1$, whereas $\mathfrak{I} \in \mathsf{LOCC}_\mathbb{N}$ if $\mathfrak{I} \in \mathsf{LOCC}_k$ for some $k \in \mathbb{N}$. Eventually, we are able to define the LOCC instrument set.

Definition 3 (LOCC)**.** The quantum instrument $\mathfrak{I}$ is an element of LOCC if there exists a sequence $\{\mathfrak{I}_n\}$ of quantum instruments in which

1. $\mathfrak{I}_0 \in \mathsf{LOCC}_\mathbb{N}$.
2. $\mathfrak{I}_n$ is LOCC linked to $\mathfrak{I}_{n-1}$.
3. Each $\mathfrak{I}_n$ has a coarse graining $\mathfrak{V}_n$ such that

$$\lim_{n\to\infty} \mathfrak{V}_n = \mathfrak{I}. \tag{2.8}$$

Remark (Quantum instrument topology)*.* The set of instruments over the same index set carries a metric as follows. For two instruments $\mathfrak{I} = \{\mathcal{T}_i : i = 1 \dots n\}$ and $\mathfrak{K} = \{\mathcal{W}_i : i = 1 \dots n\}$ we define a distance measure induced by the diamond norm $\|\cdot\|_\diamond$ on the associated quantum transformations, namely

$$D(\mathfrak{I}, \mathfrak{K}) = \sum_{i=1}^{n} \|\mathcal{T}_i - \mathcal{W}_i\|_\diamond = \max_{0 \le \rho \le I} \sum_{i=1}^{n} \|\mathcal{I} \otimes (\mathcal{T}_i - \mathcal{W}_i)(\rho)\|_1, \tag{2.9}$$

where $\|\cdot\|_1$ is the trace norm, I the identity on the Hilbert space $\mathcal{H}_Q$ and ρ a positive operator on $\mathcal{H}_Q \otimes \mathcal{H}_Q$. For the convergence of instrument sequences, as in eq. (2.8), we further assume that there exists an index $\bar{n} \in \mathbb{N}$ such that the limit $\mathfrak{I}$ and the terms $\mathfrak{V}_n$ have the same number of transformations for all $n > \bar{n}$.

We finally list some relevant properties of the LOCC instruments. We know that the sets introduced so far undergo the following chain of proper inclusions [Chi+14, Eq. (3)]

$$\mathsf{LOCC}_1 \subset \mathsf{LOCC}_k \subset \mathsf{LOCC}_{k+1} \subset \mathsf{LOCC}_{\mathbb{N}} \subset \mathsf{LOCC} \subset \mathrm{SEP}. \tag{2.10}$$

We empathize the last relation of eq. (2.10) and that even the topological closure of $\mathsf{LOCC}_{\mathbb{N}}$ is a proper subset of SEP. The set LOCC forms a convex subset of quantum instruments.

2.2 State discrimination in the quantum theory

In the branch of quantum information, we investigate the properties and feasibility of protocols for state discrimination of quantum systems. We are given a couple of states ρ, σ for a multipartite system Q and we want to distinguish between the two making use of transformations and measurements. The discrimination quest is rather pragmatic: let us consider, for instance, that we write down the simplest conceivable piece of information, that is a bit, on the quantum system Q. Then, as soon as we want to read that information, we have to distinguish between two states used for encoding. Bipartite systems $Q = AB$ are a preferred subject of study and, in particular, we are interested in the discrimination capabilities of local operations applied onto the systems separately. The results of the investigation eventually shed some light on the fundamental features of the quantum theory.

We assume the systems lying in pure states, thus we can define two normalized vectors $|\psi\rangle, |\phi\rangle \in \mathcal{H}_{AB}$ such that

$$\rho = p\,|\psi\rangle\langle\psi| \quad \text{and} \quad \sigma = q\,|\phi\rangle\langle\phi|, \quad p + q = 1 \tag{2.11}$$

where p, q are the preparation probabilities for the two states, usually assumed to be one half each. A black box provides us with the system Q either in the state ρ or σ and we look for a measurement protocol that allows us to estimate in which state the system has been prepared. At this stage, we introduce three possible working schemes depending on the initial state conditions and on the optimization strategy applied:

Perfect discrimination We require the protocol to distinguish the right state in a single-shot manner or, equivalently, to succeed with probability equal to one. The quest is feasible since the perfect discriminability axiom[1] of QT ensures us the existence of two perfectly discriminable states when they are not completely mixed. Moreover, two states are perfectly discriminable if and only if they are orthogonal, i. e. their supports are orthogonal subspaces.

Optimal conclusive discrimination If we release the orthogonality constraint, we lose the ability to perfectly discriminate two pure states. Hence, we look for the optimal discrimination strategy that minimizes the error probability p_e of identifying the wrong state. In [Hel67], Helstrom proves the existence of such a protocol for any couple of pure states, as we later show in § 2.2.2.

Optimal unambiguous discrimination The protocol returns an unequivocal result of the discrimination with error probability equal to zero. However, the trade-off is to allow for inconclusive outputs, usually labeled with a question mark, where no information is gained from the system. The thesis does not deal with unambiguous discrimination, for further details see [Che00; Che04].

Literature comprehensively studies all three discrimination strategies, along with their relations to locality constraints applied to measurements on multipartite systems.

2.2.1 The orthogonal case

We perfectly discriminate the state ρ from σ if and only if they are orthogonal, i. e. when the two vectors $|\psi\rangle$, $|\phi\rangle$ from eq. (2.11) fulfill

$$\langle\psi|\phi\rangle = 0. \tag{2.12}$$

In [Wal+00], Walgate et al. show that the discrimination protocol may always be implemented through LOCC by observing the following strategy. Let $\{|i\rangle_A\}_{i=1\dots n}$ be an orthonormal basis for Alice such that the two states are represented as

$$\begin{aligned} |\psi\rangle &= |1\rangle_A |\eta_1\rangle_B + |2\rangle_A |\eta_2\rangle_B + \cdots + |n\rangle_A |\eta_n\rangle_B\,, \\ |\phi\rangle &= |1\rangle_A \left|\eta_1^\perp\right\rangle_B + |2\rangle_A \left|\eta_2^\perp\right\rangle_B + \cdots + |n\rangle_A \left|\eta_n^\perp\right\rangle_B\,, \end{aligned} \tag{2.13}$$

where $\{|\eta_i\rangle_B\}$ and $\{\left|\eta_i^\perp\right\rangle_B\}$ are sub-normalized vectors resulting from the factorization of the Bob part. For the sake of simplicity we assume $\dim\mathcal{H}_A \le \dim\mathcal{H}_B$ and that

$$\left\langle\eta_i\middle|\eta_i^\perp\right\rangle = 0 \quad \forall i. \tag{2.14}$$

Then the LOCC discrimination is carried out as noted below: Alice simply measures her local basis $\{|i\rangle_A\}$ and sends the output i to Bob, who in turns assesses whether his part is either $|\eta_\mathrm{i}\rangle$ or $\left|\eta_\mathrm{i}^\perp\right\rangle$ and perfectly distinguishes the right state. In the following we show that a representation as in eq. (2.13) is always possible.

We start from the general representation

$$\begin{aligned} |\psi\rangle &= |1\rangle_A |\eta_1\rangle_B + |2\rangle_A |\eta_2\rangle_B + \cdots + |n\rangle_A |\eta_n\rangle_B \\ |\phi\rangle &= |1\rangle_A |\nu_1\rangle_B + |2\rangle_A |\nu_2\rangle_B + \cdots + |n\rangle_A |\nu_n\rangle_B\,, \end{aligned} \tag{2.15}$$

[1] Cf. CDP11; DCP16.

where $\{|\eta_i\rangle_B\}$ and $\{|\nu_i\rangle_B\}$ are neither normalized nor orthogonal. Bob then chooses an orthonormal basis $\{|j\rangle\}_{j=1\ldots m}$ and expresses the factorized vectors $\{|\eta_i\rangle_B\}$ and $\{|\nu_i\rangle_B\}$ in terms of the new basis:

$$|\eta_i\rangle_B = \sum_j F_{ij}\,|j\rangle_B \quad \text{and} \quad |\nu_i\rangle_B = \sum_j G_{ij}\,|j\rangle_B\,, \tag{2.16}$$

where F and G are two $n \times m$ matrices. It is now convenient to consider the resulting product matrix $FG^\dagger$, as we obtain

$$FG^\dagger = \begin{pmatrix} \langle\nu_1|\eta_1\rangle & \cdots & \langle\nu_1|\eta_n\rangle \\ \vdots & \ddots & \vdots \\ \langle\nu_n|\eta_1\rangle & \cdots & \langle\nu_n|\eta_n\rangle \end{pmatrix}, \quad \langle\nu_i|\eta_j\rangle = \sum_{k=1}^{n} F_{jk}G^*_{ik}. \tag{2.17}$$

We are now ready to prove that the perfectly discriminability of pure orthogonal states is implementable by means of LOCC.

Theorem 1 (Walgate et al.)**.** *Two orthogonal pure states*

$$\rho = p\,|\psi\rangle\langle\psi| \quad \textit{and} \quad \sigma = q\,|\phi\rangle\langle\phi|\,, \quad p+q=1 \tag{2.11}$$

are always perfectly discriminable through LOCC.

Proof. We look for a suitable basis for Alice such that the relation of eq. (2.14) is valid for the $\{|\eta_i\rangle_B\}$ and $\{|\nu_i\rangle_B\}$ vectors. The orthogonality condition of eq. (2.12) leads here to a traceless $FG^\dagger$ matrix, i. e.

$$\langle\phi|\psi\rangle = \sum_i \langle\nu_i|\eta_i\rangle = \mathrm{Tr}\big[FG^\dagger\big] = 0. \tag{2.18}$$

Thus, we are interested in (i) Understanding the behavior of the matrix $FG^\dagger$ under change of basis (ii) Proving the existence of a unitary matrix $V \in \mathsf{U}(\mathcal{H}_A)$ for Alice such that the matrix $FG^\dagger$ is zero-diagonal, leading to eq. (2.14).

Whenever Alice changes her basis by applying a unitary matrix V while Bob does the same through the matrix $W \in \mathsf{U}(\mathcal{H}_B)$, we write

$$|i\rangle_A = \sum_{i'} V^\dagger_{ii'}\,|i'\rangle_A \quad \text{and} \quad |j\rangle_B = \sum_{j'} W^\dagger_{jj'}\,|j'\rangle_B\,. \tag{2.19}$$

From eqs. (2.15) and (2.16) we have

$$|\psi\rangle = \sum_{ij} |i\rangle_A\,F_{ij}\,|j\rangle_B = \sum_{ii'jj'} |i'\rangle_A\,|j'\rangle_B\,V^\dagger_{ii'}F_{ij}W^\dagger_{jj'}, \tag{2.20}$$

thus the matrix F has the components

$$F'_{ij} = \sum_{i'j'} V^*_{i'i}F_{ij}W^\dagger_{jj'} \tag{2.21}$$

in the new bases. We evaluate the same result for the vector $|\phi\rangle$ to achieve

$$F' = V^*FW^\dagger \quad \text{and} \quad G' = V^*GW^\dagger; \tag{2.22}$$

the matrix $FG^\dagger$ has now the form $F'G'^\dagger = V^* F G^\dagger V^{*\dagger}$, V^* is a unitary matrix since V is, and the operator W disappears as expected. We define $U = V^*$ and look for the right matrix choice in order to attain zero-diagonal $UFG^\dagger U^\dagger$.

Thanks to theorem 10 in appendix A, we find a matrix U such that all diagonal components of $M = FG^\dagger$ are equal. Nevertheless, from eq. (2.18) we conclude that $FG^\dagger$ is zero-diagonal as required. Theorem 10 proves the existence of U for systems made of qubits only, though we can overcome such limitation by enlarging the quantum systems under study. One possible realization is the following: Alice introduces the ancillary qubit in the state $|0\rangle$ and the resulting state expression would then read as

$$|\psi\rangle = |1\rangle_A |0\rangle |\eta_1\rangle_B + \cdots + |n\rangle_A |0\rangle |\eta_n\rangle_B \\ + |1\rangle_A |1\rangle |\eta'_1\rangle_B + \cdots + |n\rangle_A |1\rangle |\eta'_n\rangle_B \quad (2.23)$$

where $|\eta'_i\rangle_B = 0\ \forall i$. The extended matrix $F_e G_e^\dagger$ has now the dimensions $2n \times 2n$ and, since between n and $2n$ there is always a power of two[2], we can find a proper equidiagonalizable submatrix. Alice applies the unitary matrix onto the submatrix and transforms $FG^\dagger$ as well, working out the correct measure basis. □

The theorem shows an efficient manner to carry out a protocol for perfect discrimination by only means of LOCC_1. In conclusion we understand that in QT, the set of separable effects and LOCC protocols have the same capabilities for the purpose of discriminating pairs of orthogonal states.

2.2.2 Optimal conclusive discrimination

The orthogonality constraint is necessary for perfect discrimination and, when we release it, we have to set up a discrimination strategy that may be optimized. We introduce a POVM whose terms are $\{\Pi_\psi, \Pi_\phi\}$, for measuring $|\psi\rangle$ and $|\phi\rangle$ respectively, such that $0 \leq \Pi_\psi, \Pi_\phi \leq I$ and $\Pi_\psi + \Pi_\phi = I$. The optimal conclusive measurement protocol minimizes the error probability of detecting the wrong state

$$p_{opt} = \min_{\{\Pi_\psi, \Pi_\phi\}} p_e(\Pi_\psi, \Pi_\phi) \quad (2.24)$$

where

$$p_e = \mathrm{Tr}\,[\Pi_\psi \sigma + \Pi_\phi \rho] = \langle\phi|\Pi_\psi|\phi\rangle + \langle\psi|\Pi_\phi|\psi\rangle\,. \quad (2.25)$$

The POVM does exist for any pair of pure states as first shown by [Hel67]. The protocol requires the introduction of the operator

$$\Delta = \rho - \sigma = p\,|\psi\rangle\langle\psi| - q\,|\phi\rangle\langle\phi|\,, \quad (2.26)$$

and its diagonalization

$$\Delta = \lambda_+ |+\rangle\langle+| + \lambda_- |-\rangle\langle-|\,, \quad (2.27)$$

[2]If $2^k < n < 2n < 2^{k+1}$, by dividing all terms by two we have $2^{k-1} < n/2 < n < 2^k < n$. □

where the eigenvalues respect $\lambda_+ > 0$, $\lambda_- < 0$ and $\langle +|-\rangle = 0$. The optimal discrimination strategy is to measure in the diagonalization basis and return the state $|\psi\rangle$ for the output being $|+\rangle$, $|\phi\rangle$ otherwise. At this stage, we rigorously prove the above assertion.

Theorem 2 (Helstrom)**.** *The measurment in the diagonalization basis of the operator* Δ*, as in eqs.* (2.26) *and* (2.27)*, provides the optimal conclusive discrimination strategy for the states* ρ*,* σ*.*

Proof. Let $\{\lambda_+, |+\rangle\, ; \lambda_-, |-\rangle\}$ be the spectrum of operator Δ. We write the error probability of eq. (2.25) in the diagonalization basis once we fixed the probabilities p, q:

$$\begin{aligned} p_e &= \mathrm{Tr}\left[\Pi_\psi \sigma + (I - \Pi_\psi)\rho\right] = p - \mathrm{Tr}\left[\Pi_\psi \Delta\right] \\ &= p - \left(\lambda_+ \langle +|\Pi_\psi|+\rangle + \lambda_- \langle -|\Pi_\psi|-\rangle\right). \end{aligned} \tag{2.28}$$

Since (i) the eigenvalues λ_+, λ_- are strictly positive and negative respectively, as the labels indicate, (ii) the eigenvectors are orthogonal (iii) the operators Π_ψ, Π_ϕ are positive, we achieve the optimal strategy for

$$\Pi_\psi = |+\rangle\langle +| \quad \text{and} \quad \Pi_\phi = |-\rangle\langle -|\,. \tag{2.29}$$

□

We point out that the above protocol can return a wrong result, in the sense of detecting the incorrect state in reference to the one provided by the black box. Indeed, this kind of discrimination strategy is called *conclusive*, as there is always an outcome, though *ambiguous* since it is flawed.

According to [Vir+01], the optimal detection protocol of Helstrom, once the operator Δ has been correctly diagonalized in the basis $\{|+\rangle\, , |-\rangle\}$, reduces to the discrimination of two orthogonal states. Nevertheless, in the last section § 2.2.1 we showed that perfect discrimination is implementable through only LOCC thanks to theorem 1. Since the measurement is between the two vectors $\{|+\rangle\, , |-\rangle\}$, we prove that the optimal conclusive discrimination strategy in the quantum theory is implementable by only means of LOCC.

Chapter 3

The Fermionic Quantum Theory

In 1984, R. Feynman wonders whether it is possible to simulate the behavior of Fermionic systems through quantum qubits[1]. Since then, the properties of Fermions have been thoroughly investigated both in terms of computational capabilities and operational features. On the one hand, the former aspect sheds light on the underlying informational structure of the theory, with the striking result that the Fermionic theory and quantum theory of qubits are computationally equivalent, as proved by [BK02]. On the other hand, the latter leads to a deeper understanding of the physical traits of the Fermionic theory, especially to the notions of locality and entanglement.

Fermions are half-integer spin particles that undergo the Pauli exclusion principle, i.e. two Fermionic particles cannot occupy the same state at the same time. We present the theory in the second quantization formalism as a superselection of the quantum one. In particular, we treat the Fermionic quantum theory (FQT) as an OPT having local Fermionic modes as elementary systems, which represents the counterpart of qubits in quantum theory. From the computational sense a local Fermionic mode is a system which can be either empty or occupied by a single "excitation." Within this framework, the Fermionic parity superselection rule as been derived, see [DAr+14], as a consistency constraint of the Fermionic probabilistic theory. Eventually, we describe some correspondences to the quantum theory through the Jordan-Wigner transformation.

3.1 The Fermionic algebra

The notion of locality in the FQT is rigorously defined through the Fermionic algebra $\mathfrak{F}$. For n local Fermionic modes, we consider the annihilation and creation operators a_i, $a_i^\dagger$ as those satisfying the canonical anticommutation relations

$$\left\{a_i, a_j^\dagger\right\} = \delta_{ij} I \quad \text{and} \quad \{a_i, a_j\} = 0, \tag{3.1}$$

where $i, j = 1 \dots n$. We further inspect the properties of the underlying Hilbert space if we introduce the number operators as $N_i = a_i^\dagger a_i$. From the anticommu-

[1] Fey82.

tation relations of eq. (3.1), we conclude that their spectrum is

$$\sigma(N_i) = \{0, |0\rangle_i ; 1, |1\rangle_i\}, \tag{3.2}$$

i. e. $a_i^\dagger a_i |0\rangle_i = 0$ and $a_i^\dagger a_i |1\rangle_i = |1\rangle_i$. Moreover, the annihilation and creation operators a_i, $a_i^\dagger$ satisfy the relevant properties of

$$a_i |1\rangle_i = |0\rangle_i , \tag{3.3}$$

$$a_i^\dagger |0\rangle_i = |1\rangle_i , \tag{3.4}$$

due to eq. (3.1).

The Fermionic operators a_i, $a_i^\dagger$ are the generator of the Fermionic algebra $\mathfrak{F}(n)$ for n local Fermionic modes while eqs. (3.3) and (3.4) allow us to interpret them as lowering and raising operators, respectively, for the number operator $a_i^\dagger a_i$. Furthermore, the operators $a_i^\dagger a_i$ mutually commute and are simultaneously diagonalizable. We define the vacuum state $|\Omega\rangle$ as the unique shared eigenvector whose eigenvalue is equal to zero, namely

$$a_i^\dagger a_i |\Omega\rangle = 0 \quad \forall i = 1 \dots n. \tag{3.5}$$

Starting from the vacuum vector $|\Omega\rangle$ and accordingly applying the creation operators $a_i^\dagger$, we introduce the Fock basis of elements

$$|s_1 s_2 \dots s_n\rangle = (a_1^\dagger)^{s_1} (a_2^\dagger)^{s_2} \cdots (a_n^\dagger)^{s_n} |\Omega\rangle , \quad s_i = 0, 1 \tag{3.6}$$

which spans the antisymmetric Fock space

$$\mathcal{F}^n = \mathrm{Span}_{\mathbb{R}}\{|s_1 s_2 \dots s_n\rangle : s_i = 0, 1\} \tag{3.7}$$

of dimension equal to 2^n. The term s_i is the occupation number of the i-th mode and corresponds to the expectation value of the number operator $a_i^\dagger a_i$. Finally, we point out that a vector of eq. (3.6) represents a Slater determinant in the first quantization formalism.

3.2 The parity superselection rule

We briefly introduce the Fermionic quantum theory starting from the assumption made in [DAr+14], so that we can infer its most relevant properties. The theory deals with systems made of the composition of local Fermionic modes and is derived starting from the superselected states of the quantum theory of qubits. We then require the atomic and local transformations to act on their systems through the Fermionic operators of the algebra $\mathfrak{F}$ we introduced above. The FQT manifests new and distinctive traits ranging from a different structure of state and effect sets to an alternative notion of entanglement. We begin by assuming the following postulates:

1. The Fermionic quantum theory is causal.
2. The states of n local Fermionic modes are represented by density matrices on the antisymmetric Fock space $\mathcal{F}^n$.

3. The transformations on n local Fermionic modes are represented by linear Hermitian-preserving maps.

4. For a composite system $Q = AB$ of n modes, the local transformations on the subsystem A of the first $1 \dots m < n$ modes have Kraus operators generated by the Fermionic operators $a_j, a_j^\dagger \in \mathfrak{F}(m)$ for $j = 1 \dots m$.

5. Local transformations on a system retain the same Kraus representation when other systems are added or discarded.

6. The transformation of Kraus operators $X_i = a_i + a_i^\dagger \ \forall i = 1 \dots n$ is physical, namely it is an admissible map of the theory.

7. The paring between states and effects is given by the Born rule
$$p = (a|\rho) = \mathrm{Tr}[a\rho]. \tag{3.8}$$

8. On a single mode the pairing between the deterministic effect e and the state ρ is $(e|\rho) = \mathrm{Tr}[\rho]$.

The Fermionic algebra takes here the crucial role of defining the locality of transformations. Indeed, in assumption 4 we require the Kraus operator of an atomic and local transformation to belong to the algebra of the Fermionic modes the map is acting upon. Moreover, assumptions 4 and 6 let us derive a relevant property of any transformation between n local Fermionic modes, namely that each Kraus operator is a combination of either an even or odd number of field operators.

The previous results lead to the following two fundamental features of the Fermionic theory, which allow us to characterize both the sets of preparations and effects.

Theorem 3 (D'Ariano et al.)**.** *States of* FQT *satisfy the parity superselection rule, i. e. their density matrices commute with the parity operator*
$$P = \frac{1}{2}\left[I + \prod_{i=0}^{n}\left(a_i a_i^\dagger - a_i^\dagger a_i\right)\right]. \tag{3.9}$$

Theorem 4 (D'Ariano et al.)**.** *Effects of the* FQT *are positive operators made of products of an even number of fields operators.*

The former theorem restricts the set of possible pure states for Fermionic systems only to those having a well-defined parity. The antisymmetric Fock space of n modes decomposes into the direct sum
$$\mathcal{F}^n = \mathcal{F}_0^n \oplus \mathcal{F}_1^n, \tag{3.10}$$
the subscript indicating the eigenvalue of the parity operator, and the set of states also decomposes as well in
$$\mathrm{St}(\mathcal{F}^n) = \mathrm{St}(\mathcal{F}_0^n) \oplus \mathrm{St}(\mathcal{F}_1^n). \tag{3.11}$$
If we consider vectors in the form of eq. (3.6), the parity is the sum of the excitations modulo two
$$s = \sum_{i=0}^{n} s_i \mod 2 \tag{3.12}$$

or, equivalently, whether the total occupation number $s = \sum_i s_i$ is even or odd.

We point out that the set structure of Fermionic states is strongly shaped by the parity superselection rule and irreversibly altered from the starting quantum one. Since convex-only combinations between vectors of different parity are allowed, the particular case of a single isolated mode surprisingly reduces to the classical bit

$$\mathrm{St}(\mathcal{F}^1) = \{p\,|0\rangle\langle 0| + (1-p)\,|1\rangle\langle 1| : p \in [0,1]\}. \tag{3.13}$$

Beside, the whole set of states $\mathrm{St}(\mathcal{F}^n)$ is spanned by the convex combinations of the even and odd preparations as depicted in fig. 3.1, where the case of two local Fermionic modes is considered. For two modes, the even and odd states separately have the supports lying on a bidimensional space, shown as two Bloch spheres in the figure. The states represented on the spheres are pure, whereas those inside the balls and the convex combination between them are the Fermionic mixed states.

Generally, the vector space of parity-defined vectors is isomorph to that of $n-1$ qubits, where n is the number of local Fermionic modes. On the other hand, the linear span of states and effects corresponds to the space of $2^n \times 2^n$ hermitian matrices

$$\mathrm{St}_{\mathbb{R}}(\mathcal{F}^n) = \mathrm{Eff}_{\mathbb{R}}(\mathcal{F}^n) = \mathrm{Herm}((\mathbb{C}^2)^{\otimes n}), \tag{3.14}$$

whose dimension is 2^{2n}. Once we reordered the Fock basis so that the even vectors precede the odd ones, we obtain $\forall \rho \in \mathrm{St}(n)$ and $\forall a \in \mathrm{Eff}(n)$ that

$$\rho = \left(\begin{array}{c|c} \rho_0 & \\ \hline & \rho_1 \end{array}\right), \qquad \rho_0, \rho_1 \geq 0 \quad \text{and} \quad \mathrm{Tr}[\rho_0 + \rho_1], \leq 1 \tag{3.15}$$

$$a = \left(\begin{array}{c|c} a_0 & \\ \hline & a_1 \end{array}\right), \qquad 0 \leq a_0 \leq I_0 \quad \text{and} \quad 0 \leq a_1 \leq I_1, \tag{3.16}$$

namely, the preparations and effects are represented as block matrices on the even and odd subspaces $\mathcal{F}_0$, $\mathcal{F}_1$.

3.3 Jordan-Wigner transformation

We further understand the locality and entanglement features of the FQT only once we introduce the Jordan-Wigner transformation between local Fermionic modes and quantum qubits, firstly proposed in [JW28]. The antisymmetric Fock space $\mathcal{F}^n$ is isometric to the complex Hilbert space of n qubits, as we promptly realize by looking at the Fock basis of eq. (3.6), and let us define the unitary map

$$\begin{aligned} U\colon \mathcal{F}^n &\to \mathbb{C}^{2^n} \\ |s_1 s_2 \ldots s_n\rangle_F &\mapsto |s_1 s_2 \ldots s_n\rangle_Q . \end{aligned} \tag{3.17}$$

Given the Pauli matrices

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.18}$$

we could then promote the operators

$$\sigma_i^{\pm} = \frac{\sigma_i^x \pm i\sigma_i^y}{2} \tag{3.19}$$

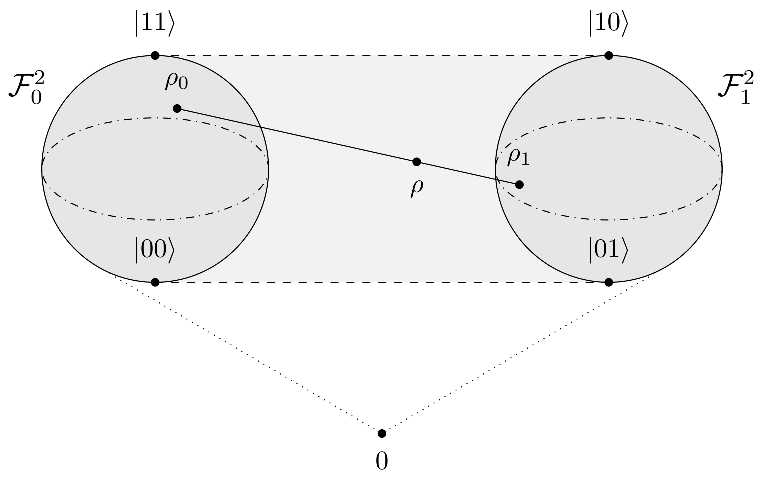

Figure 3.1: A pictorial representation of the state set for two local Fermionic modes is shown. The shaded surface depicts the set of deterministic states, whereas the underlying area delimited by the dotted lines converging to the zero state refers to the conic structure of sub-deterministic preparations. The represented states satisfy $\operatorname{Supp}\rho_s \subseteq \mathcal{F}_s^2$ for $s = 0, 1$, while $\rho = p\rho_0 + (1-p)\rho_1$, $p \in [0, 1]$ is any convex combination of the previous two.

to the quantum equivalents of the creation and annihilation Fermionic operators. On the one hand, we have the correct same-site anticommutation relations $\left\{\sigma_i^+, \sigma_i^-\right\} = I$, $i = 1 \dots n$. However, on the other hand, spins on different sides commute, since $\left[\sigma_i^+, \sigma_j^-\right] = 0$, unlike Fermions which anticommute. We accordingly correct our operators by adding a phase factor able to keep track of the other excited modes, attaining the Jordan-Wigner transformation

$$\mathcal{J}(a_i) = e^{-i\pi\sum_{j=1}^{i-1}\sigma_j^+\sigma_j^-} \cdot \sigma_i^- \tag{3.20}$$

$$\mathcal{J}(a_i^\dagger) = e^{+i\pi\sum_{j=1}^{i-1}\sigma_j^+\sigma_j^-} \cdot \sigma_i^+ \tag{3.21}$$

$$\mathcal{J}(a_i^\dagger a_i) = \sigma_i^+\sigma_i^-, \tag{3.22}$$

while the phase term may also be written as

$$e^{\pm i\pi\sum_{j=1}^{i-1}\sigma_j^+\sigma_j^-} = \prod_{j=1}^{i-1} e^{\pm i\pi\sigma_j^+\sigma_j^-} = \prod_{j=1}^{i-1}(1 - 2\sigma_j^+\sigma_j^-) = \prod_{j=1}^{i-1}(-\sigma_j^z). \tag{3.23}$$

The transformation $\mathcal{J}$ is actually a $*$-algebra isomorphism and let us build a Fermionic algebra $\mathfrak{F}(n)$ on the top of a n qubits system. We may be tempted to translate all Fermionic expressions into quantum ones through the Jordan-Wigner transformation, however many notion of QT will not apply once they are transformed back. For instance, as we can see from eqs. (3.20) and (3.21), local Fermionic operators are generally mapped to a many qubits operator. Therefore, the locality properties are preserved only if the expression involves an even number of Fermionic operators for each site, in order to cancel the phase factor as in eq. (3.22). For an extensive addendum on the Jordan-Wigner transformation see [Nie05].

Finally, we conclude by showing a pertinent result for the succeeding chapter 4, namely that Fermionic LOCC correspond to quantum ones.

Theorem 5 (D'Ariano et al.)**.** *Every Fermionic LOCC corresponds to a quantum LOCC on qubits under the Jordan-Wigner transformation.*

All the tools required for the study of LOCC-discrimination in the Fermionic theory have been presented. We anticipate that in chapter 4 all expressions deal with an even number of Fermionic operators, hence we are allowed to carefree work with the quantum notation we are used to thanks to the Jordan-Wigner transformation.

Chapter 4

Pure State Discrimination in the Fermionic Theory

In this chapter, we investigate the protocols for pure state discrimination in the Fermionic theory. For two bipartite states, we examine in detail the criteria for the implementation through LOCC of orthogonal and optimal non-orthogonal state discrimination. Moreover, we show that it is always possible to perfectly discriminate two orthogonal pure preparations using LOCC by taking advantage of a shared entangled resource between the two parts. The results of chapter 2 are of great use, since in particular circumstances the Fermionic states behave the same way as quantum ones. In the most general approach, we assume that Alice holds n local Fermionic modes whereas Bob m of them. Let $|\psi\rangle\,,\ |\phi\rangle \in \mathcal{F}^{n+m}$ and $\rho = |\psi\rangle\langle\psi|\,,\ \sigma = |\phi\rangle\langle\phi|$ be two bipartite pure states. We show readily that if they feature different parity, e. g. $|\psi\rangle \in \mathcal{F}_0^{n+m}$ and $|\phi\rangle \in \mathcal{F}_1^{n+m}$, they are discriminable only by means of LOCC. Alice has to locally measure P_0 and P_1, i. e. the projectors on $\mathcal{F}_0^n$ e $\mathcal{F}_1^n$ respectively, while Bob carries out the same measurement to his system. After that, they tell each other the result through a classical channel and whenever they read the same outcome, the state belongs to the even parity subspace $\mathcal{F}_0^{n+m}$, it lays in $\mathcal{F}_1^{n+m}$ otherwise.

Once we handled the previous case, we focus our study on the discrimination of two preparation that have the same parity. As it is always possible to swap between even and odd states just by means of local reversible transformation, i. e. unitary matrices, all results dealing with LOCC and even states are valid for the odd ones too. Therefore, we introduce the following convenient notation to describe even states $|\psi\rangle \in \mathcal{F}_0^{n+m}$:

$$|\psi\rangle = \psi_E\,|\psi_E\rangle + \psi_O\,|\psi_O\rangle = \psi_E \sum_{i=1}^{c} |E_i\rangle_A \left|\eta_i^E\right\rangle_B + \psi_O \sum_{j=1}^{c} |O_j\rangle_A \left|\theta_j^O\right\rangle_B\,, \quad (4.1)$$

where $c = \min(n, m)$ and

$|\psi_E\rangle \in \mathcal{F}_0^n \otimes \mathcal{F}_0^m$ is the even part of $|\psi\rangle$, viz. the projection of the vector onto the subspace of states featuring even parity on both Alice and Bob systems separately.

$\psi_E \in \mathbb{C}$ is even part amplitude.

$\{|E_i\rangle_A\}_{i=1\ldots n}$ is an orthonormal basis of even states for Alice.

$\{|\eta_i^E\rangle_B\}_{i=1\ldots c}$ is a set of even vectors resulting from the factorization of the even part $|\psi_E\rangle$, as determined by the choice of Alice's basis. In general, they are neither orthogonal nor normalized.

The same definitions correspondingly apply for the odd part of $|\psi\rangle$, once the necessary changes have been made. For the sake of clarity, we indicate with the subscript letters E, O the quantities of states for which both Alice and Bob parts are even and odd, respectively. Since $\langle\psi_E|\psi_O\rangle = 0\ \forall\, |\psi\rangle \in \mathcal{F}_0$ we have the further property that

$$|\psi_E|^2 + |\psi_O|^2 = 1. \tag{4.2}$$

For example, consider

$$|\psi\rangle = \psi_E\Bigg(\underbrace{|00\rangle}_{|E_0\rangle}\underbrace{\sqrt{2/3}\,|00\rangle}_{|\eta_0^E\rangle} + \underbrace{|11\rangle}_{|E_1\rangle}\underbrace{1/\sqrt{3}\,|11\rangle}_{|\eta_1^E\rangle}\Bigg) + \psi_O\Bigg[\underbrace{|01\rangle}_{|O_0\rangle}\underbrace{\Big(1/\sqrt{3}\,|01\rangle + i/\sqrt{3}\,|10\rangle\Big)}_{|\theta_0^O\rangle} + \underbrace{|10\rangle}_{|O_1\rangle}\underbrace{\Big(-1/\sqrt{3}\,|01\rangle\Big)}_{|\theta_1^O\rangle}\Bigg].$$

4.1 Orthogonal states

In the first part of this chapter, we show that some requirements arise for local discrimination of two orthogonal states to be achieved. Contrary to the QT, not all Fermionic states are perfectly discriminable through LOCC. Following our notation, we introduce the second state

$$|\phi\rangle = \phi_E \sum_{i=1}^{c} |E_i\rangle_A\,|\nu_i^E\rangle_B + \phi_O \sum_{j=1}^{c} |O_j\rangle_A\,|\mu_j^O\rangle_B \tag{4.3}$$

and require that

$$\langle\psi|\phi\rangle = \bar{\psi}_E\phi_E\,\langle\psi_E|\phi_E\rangle + \bar{\phi}_O\psi_O\,\langle\psi_O|\phi_O\rangle = 0. \tag{4.4}$$

In the following, we show that the LOCC discrimination of $|\psi\rangle$ and $|\phi\rangle$ is possible if the even $|\psi_E\rangle$, $|\phi_E\rangle$ and odd $|\psi_O\rangle$, $|\phi_O\rangle$ parts of the states are LOCC-distinguishable separately. We first notice that the condition of eq. (4.4) is fulfilled if the even-odd amplitudes ψ_E, ϕ_E, ψ_O and ϕ_O or the *braket* terms are equal zero. The possible cases are then grouped as follows:

- Two amplitudes are zero.

  (a) The two amplitudes are of the same type, e. g. $\psi_E = \phi_E = 0$, thus the orthogonal parts, $|\psi\rangle = |\psi_O\rangle$ and $|\phi\rangle = |\phi_O\rangle$, behave exactly as two ordinary quantum states. The condition of eq. (4.4) translates into $\langle\psi_O|\phi_O\rangle = 0$ and, thanks to Walgate et al. protocol of § 2.2.1, Alice is able to locally select her own odd basis in order to let Bob perfectly discriminate the two states when

  $$\langle\theta_i^O|\mu_i^O\rangle = 0 \quad \forall i.$$

(b) The two amplitudes are of different type, for instance $\psi_E = \phi_O = 0$, $|\psi\rangle = |\psi_O\rangle$ and $|\phi\rangle = |\phi_E\rangle$. The states $|\psi\rangle$ and $|\phi\rangle$ are of different parity, therefore perfectly discriminable through local measurements. The protocol is straightforward: Alice or Bob simply have to measure the parity locally and the correct state is recognized.

- Only one amplitude is zero. Without loss of generality, let $\psi_O = 0$, hence $|\psi\rangle = |\psi_E\rangle$ and $|\phi\rangle = \phi_E |\phi_E\rangle + \phi_O |\phi_O\rangle$. If Alice locally measures the parity of her system, she reads odd for the state being $|\phi\rangle$. Otherwise Alice and Bob locally implements the Walgate et al. protocol once again in order to distinguish

$$\sum_i |E_i\rangle_A \left|\eta_i^E\right\rangle_B \quad \text{from} \quad \phi_E \sum_{i=1}^{c} |E_i\rangle_A \left|\nu_i^E\right\rangle_B .$$

  The quest is feasible as eq. (4.4) becomes $\langle\psi_E|\phi_E\rangle = 0$ and, by the correct even basis choice,

$$\left\langle\eta_i^E\middle|\nu_i^E\right\rangle = 0 \quad \forall i.$$

- All amplitudes are different from zero, the case is investigated henceforth.

Here we have two possibilities, in the first case the even and odd parts are separately orthogonal[1], that is

$$\langle\psi_E|\phi_E\rangle = \langle\psi_O|\phi_O\rangle = 0. \tag{4.5}$$

The expression in eq. (4.5) fulfills the conditions of theorem 1 for the even and odd parts separately and it is possible to achieve

$$\left\langle\eta_i^E\middle|\nu_i^E\right\rangle = \left\langle\theta_j^O\middle|\mu_j^O\right\rangle = 0 \quad \forall i, j. \tag{4.6}$$

Namely, we can treat the two direct sum subspaces $(\mathcal{F}_0^n \otimes \mathcal{F}_0^m) \oplus (\mathcal{F}_1^n \otimes \mathcal{F}_1^m)$ as those representing two independent quantum system, to which we singly apply the discrimination protocol through LOCC. Let $\left\{|E_k'\rangle_B\right\}_{k=1\dots m}$ and $\left\{|O_k'\rangle_B\right\}_{k=1\dots m}$ be two even and odd basis for Bob, respectively. There exists two $2s \times 2m$ matrices F and G such that

$$\left|\eta_i^E\right\rangle_B = \sum_{k=1}^{m} F_{ik} |E_k'\rangle \qquad \left|\theta_j^O\right\rangle = \sum_{k=1}^{m} F_{s+j,m+k} |O_k'\rangle$$

$$\left|\nu_i^E\right\rangle = \sum_{k=1}^{m} G_{ik} |E_k'\rangle \qquad \left|\mu_j^O\right\rangle = \sum_{k=1}^{m} G_{s+j,m+k} |O_k'\rangle ,$$

and

$$\left(FG^\dagger\right)_{ab} = \begin{pmatrix} \left\langle\nu_a^E\middle|\eta_b^E\right\rangle & \left\langle\nu_a^E\middle|\theta_{b-s}^O\right\rangle = 0 \\ \left\langle\mu_{a-s}^O\middle|\eta_b^E\right\rangle = 0 & \left\langle\mu_{a-s}^O\middle|\theta_{b-s}^O\right\rangle \end{pmatrix} = \begin{pmatrix} E & 0 \\ 0 & O \end{pmatrix}. \tag{4.7}$$

This is the equivalent Fermionic representation of eq. (2.17), expressed as the direct sum of even and odd subspaces. The orthogonality condition of eq. (4.4) imply that $\mathrm{Tr}\left[FG^\dagger\right] = 0$ in the Fermionic case as well, but only the further requirement of eq. (4.5) ensures that the two blocks E and O are traceless separately.

[1]N.B.: $\langle\psi_E|\phi_E\rangle = 0 \Leftrightarrow \langle\psi_O|\phi_O\rangle = 0$ for $\psi_E, \psi_O, \phi_E, \phi_O \neq 0$.

In the second case we consider the states for which

$$\langle\psi_E|\phi_E\rangle\,,\langle\psi_O|\phi_O\rangle \neq 0. \tag{4.8}$$

Indeed, not all orthogonal Fermionic vectors satisfy eq. (4.5), consider for instance the two vectors

$$|\Psi_\pm\rangle = |00\rangle_A\,|00\rangle_B \pm |01\rangle_A\,|01\rangle_B\,. \tag{4.9}$$

They are actually orthogonal, but the even and odd parts are not perfectly discriminable separately. However, we now prove that only in the case where eq. (4.5) is satisfied, two orthogonal Fermionic states can be perfectly discriminated through LOCC. In fact, no unitary matrix $U = U^E + U^O$ is capable to equidiagonalize $FG^\dagger$ and any non-trivial transformation on the whole $\mathcal{F}_0^{n+m}$ space would inevitably act on superpositions of $|\psi_E\rangle$, $|\psi_O\rangle$ and $|\phi_E\rangle$, $|\phi_O\rangle$, leading to non-local effects. In order to rigorously prove the above assertion, we show that it is not possible to perfectly discriminate two state of this kind through separable effects SEP, thus neither by means of LOCC.

Theorem 6. *Let $\rho = |\psi\rangle\langle\psi|$ and $\sigma = |\phi\rangle\langle\phi|$ be two pure, deterministic and orthogonal states, then the following statements are equivalent:*

1. *The two states are perfectly discriminable through SEP.*
2. *The two states are perfectly discriminable through LOCC.*
3. *The even and odd parts are separately orthogonal, i. e.*

$$\langle\psi_E|\phi_E\rangle = \langle\psi_O|\phi_O\rangle = 0. \tag{4.5}$$

Before we proceed with the theorem proof, we make a brief digression on separable effects in the Fermionic theory. From assumption 7 of § 3.2 and thanks to theorem 4, we know that effects are linear functionals $(a| = \mathrm{Tr}\,[S\cdot]$ where S is a positive operator made of products of an even number of field operators. If we require all preparations being as in eqs. (4.1) and (4.3), viz. even, we can focus only on those effects actually observable. The further condition of S being separable allows us to express the operator as the direct sum

$$S = S_E + S_O \tag{4.10}$$

where

$$S_E = \sum_{ij} p_{ij}\ e_i \otimes e'_j \quad \text{and} \quad S_O = \sum_{kl} q_{kl}\ o_k \otimes o'_l. \tag{4.11}$$

In order to represent a Fermionic effect, the operator S must satisfy $0 \leq S \leq I$ while $0 \leq p_{ij}, q_{kl} \leq 1$, the operators e_i, e'_j, o_k, o'_l are positive and

$$\begin{aligned} \mathrm{Supp}(e_i) &\subseteq \mathcal{F}_0^n \qquad & \mathrm{Supp}(e'_j) &\subseteq \mathcal{F}_0^m \\ \mathrm{Supp}(o_k) &\subseteq \mathcal{F}_1^n \qquad & \mathrm{Supp}(o'_l) &\subseteq \mathcal{F}_1^m. \end{aligned}$$

Instead, had we assumed the states to be odd, the separable effect would have the form

$$S = \sum_{ij} p_{ij}\ e_i \otimes o'_j + \sum_{kl} q_{kl}\ o_k \otimes e'_l.$$

In both cases, mixed terms like e. g. $|\psi_E\rangle\langle\psi_O|$ vanish in all expectation values. We are now ready to prove that state discriminability is implementable through LOCC if and only if even and odd parts are orthogonal separately, as expressed in eq. (4.5).

Proof. (Theorem 6) It is trivial to see that *2* ⇒ *1*, whereas we have already shown above that *3* ⇒ *2* thanks to theorem 1. We now focus on the implication *1* ⇒ *3* and wonder which separable effect S maximizes the expression[2]

$$p = \sup_{S \in \mathrm{SEP}} \mathrm{Tr}\left[(\rho - \sigma)S\right]. \tag{4.12}$$

From eqs. (4.1) and (4.3) we obtain

$$\begin{aligned}\rho - \sigma = {} & |\psi_E|^2 |\psi_E\rangle\langle\psi_E| + |\psi_O|^2 |\psi_O\rangle\langle\psi_O| + \psi_E\bar{\psi}_O |\psi_E\rangle\langle\psi_O| + \bar{\psi}_E\psi_O |\psi_O\rangle\langle\psi_E| \\ & - |\phi_E|^2 |\phi_E\rangle\langle\phi_E| - |\phi_O|^2 |\phi_O\rangle\langle\phi_O| - \phi_E\bar{\phi}_O |\phi_E\rangle\langle\phi_O| - \bar{\phi}_E\phi_O |\phi_O\rangle\langle\phi_E|,\end{aligned}$$

to be substituted along with eq. (4.10) into eq. (4.12) to attain

$$\begin{aligned}p = \mathrm{Tr}\Big[\Big(& |\psi_E|^2 |\psi_E\rangle\langle\psi_E| + |\psi_O|^2 |\psi_O\rangle\langle\psi_O| - |\phi_E|^2 |\phi_E\rangle\langle\phi_E| - |\phi_O|^2 |\phi_O\rangle\langle\phi_O|\Big) \\ & \cdot (S_E + S_O)\Big].\end{aligned} \tag{4.13}$$

If there exists a separable effect $S \in \mathrm{SEP}(AB)$ such that $p = 1$ in eq. (4.12), then we have

$$\mathrm{Tr}\left[\rho S\right] = |\psi_E|^2 \langle\psi_E|S_E|\psi_E\rangle + |\psi_O|^2 \langle\psi_O|S_O|\psi_O\rangle = 1 \tag{4.14}$$

and

$$\mathrm{Tr}\left[\sigma S\right] = |\phi_E|^2 \langle\phi_E|S_E|\phi_E\rangle + |\phi_O|^2 \langle\phi_O|S_O|\phi_O\rangle = 0. \tag{4.15}$$

All expectation values of the separable effects in the even and odd part states are positive values smaller than one. Equations (4.14) and (4.15) are true if and only if

$$\langle\psi_E|S_E|\psi_E\rangle = \langle\psi_O|S_O|\psi_O\rangle = 1 \quad \text{and} \quad \langle\phi_E|S_E|\phi_E\rangle = \langle\phi_O|S_O|\phi_O\rangle = 0 \tag{4.16}$$

which are equivalent to

$$\mathrm{Tr}\left[\left(|\psi_E\rangle\langle\psi_E| - |\phi_E\rangle\langle\phi_E|\right)S_E\right] = 1 \quad \text{and} \quad \mathrm{Tr}\left[\left(|\psi_O\rangle\langle\psi_O| - |\phi_O\rangle\langle\phi_O|\right)S_O\right] = 1. \tag{4.17}$$

Hence, it is possible to perfectly discriminate the two states through separable effects, viz. $p = 1$ in eq. (4.12), only if the even and odd parts are perfectly discriminable separately, as required in eq. (4.5). □

[2] Equation (4.12) is derived from the operational norm $\|\rho\| = \sup_{\{a,b\}}(a - b|\rho)$ for deterministic theories.

4.2 Entangled assisted discrimination

The previous result seems to prevent us to any extent from discriminating two states that do not satisfy eq. (4.5). Nevertheless, it may be possible to perfectly discriminate via LOCC two pure, deterministic and orthogonal states which fulfill $\langle\psi_E|\phi_E\rangle \neq 0$ and $\langle\psi_O|\phi_O\rangle \neq 0$ by making use of an ancillary system, namely an entangled state share by Alice and Bob. Let us take the preparation

$$|\omega\rangle_{AB} = a\,|00\rangle + b\,|11\rangle \quad a, b \neq 0 \tag{4.18}$$

and consider the two extended states

$$\rho' = |\psi\rangle\langle\psi| \otimes |\omega\rangle\langle\omega| = |\psi'\rangle\langle\psi'| \quad \text{and} \quad \sigma' = |\phi\rangle\langle\phi| \otimes |\omega\rangle\langle\omega| = |\phi'\rangle\langle\phi'|\,. \tag{4.19}$$

Their respective vectors may be factorized as follows

$$\begin{aligned} |\psi'\rangle = \psi'_E & \left(\frac{a\psi_E}{\psi'_E} \sum_{i=0}^{s} |E_i 0\rangle_A \left|\eta_i^E 0\right\rangle_B + \frac{b\psi_O}{\psi'_E} \sum_{j=0}^{s} |O_j 1\rangle_A \left|\theta_j^O 1\right\rangle_B \right)_E \\ & + \psi'_O \left(\frac{b\psi_E}{\psi'_O} \sum_{i=0}^{s} |E_i 1\rangle_A \left|\eta_i^E 1\right\rangle_B + \frac{a\psi_O}{\psi'_O} \sum_{j=0}^{s} |O_j 0\rangle_A \left|\theta_j^O 0\right\rangle_B \right)_O \end{aligned} \tag{4.20}$$

where $|\psi'_E|^2 = |a\psi_E|^2 + |b\psi_O|^2$ and $|\psi'_O|^2 = |b\psi_E|^2 + |a\psi_O|^2$, and

$$\begin{aligned} |\phi'\rangle = \phi'_E & \left(\frac{a\phi_E}{\phi'_E} \sum_{i=0}^{s} |E_i 0\rangle_A \left|\nu_i^E 0\right\rangle_B + \frac{b\phi_O}{\phi'_E} \sum_{j=0}^{s} |O_j 1\rangle_A \left|\mu_j^O 1\right\rangle_B \right)_E \\ & + \phi'_O \left(\frac{b\phi_E}{\phi'_O} \sum_{i=0}^{s} |E_i 1\rangle_A \left|\nu_i^E 1\right\rangle_B + \frac{a\phi_O}{\phi'_O} \sum_{j=0}^{s} |O_j 0\rangle_A \left|\mu_j^O 0\right\rangle_B \right)_O \end{aligned} \tag{4.21}$$

where $|\phi'_E|^2 = |a\phi_E|^2 + |b\phi_O|^2$ and $|\phi'_O|^2 = |b\phi_E|^2 + |a\phi_O|^2$. We now evaluate the scalar product

$$\langle\psi'|\phi'\rangle = \bar{\psi}'_E \phi'_E \langle\psi'_E|\phi'_E\rangle + \bar{\psi}'_O \phi'_O \langle\psi'_O|\phi'_O\rangle = \underbrace{\langle\psi|\phi\rangle}_{0} \underbrace{\langle\omega|\omega\rangle}_{1} = 0 \tag{4.22}$$

and, from eqs. (4.20) and (4.21), those of the new even and odd parts

$$\bar{\psi}'_E \phi'_E \langle\psi'_E|\phi'_E\rangle = \bar{\psi}_E \phi_E |a|^2 \langle\psi_E|\phi_E\rangle + \bar{\psi}_O \phi_O |b|^2 \langle\psi_O|\phi_O\rangle \tag{4.23}$$

$$\bar{\psi}'_O \phi'_O \langle\psi'_O|\phi'_O\rangle = \bar{\psi}_E \phi_E |b|^2 \langle\psi_E|\phi_E\rangle + \bar{\psi}_O \phi_O |a|^2 \langle\psi_O|\phi_O\rangle\,. \tag{4.24}$$

With the following theorem, we show that by means of a *maximally entangled* ancilla we are actually able to discriminate any couple of orthogonal states only through LOCC.

Theorem 7. *Let $\rho = |\psi\rangle\langle\psi|$ and $\sigma = |\phi\rangle\langle\phi|$ be two pure, deterministic and orthogonal states, it is always possible to perfectly and locally discriminate between the two preparations via LOCC if we take advantage of an ancillary system prepared in the state*

$$|\omega\rangle = \frac{1}{\sqrt{2}} \left(|00\rangle + e^{i\varphi}\,|11\rangle \right), \quad \varphi \in [0, 2\pi]. \tag{4.25}$$

Proof. From eqs. (4.22) to (4.24) and for $|a|^2 = |b|^2 = \frac{1}{2}$ we have that

$$\bar{\psi}'_E \phi'_E \langle \psi'_E | \phi'_E \rangle = \bar{\psi}'_O \phi'_O \langle \psi'_O | \phi'_O \rangle = \frac{1}{2} \left(\bar{\psi}_E \phi_E \langle \psi_E | \phi_E \rangle + \bar{\phi}_O \psi_O \langle \psi_O | \phi_O \rangle \right)$$
$$= \frac{1}{2} \langle \psi | \phi \rangle = 0, \quad (4.26)$$

which leads to eq. (4.5). We are now able to apply the protocol of Walgate et al. to the new states as shown in the previous section. □

Finally, we point out that the state provided in eq. (4.18) is actually general, as it enables the discrimination of two odd states in $\mathcal{F}_1^{n+m}$ too.

4.3 Optimal discrimination

If we release the orthogonality condition, we lose the ability to perfectly discriminate two states, as it happens in the quantum theory. The quest is firstly to introduce a protocol that reduces the error probability of detecting the wrong state and, secondly, to assess whether it is implementable through LOCC. As long as we do not require any locality constraint, the optimal protocol exists and is the quantum one, as proposed in [Hel67]. Given two deterministic and pure states $\rho = |\psi\rangle\langle\psi|$, $\sigma = |\phi\rangle\langle\phi|$ with two probabilities $p + q = 1$ and[3] $\alpha = \langle \psi | \phi \rangle \in \mathbb{R}$, we introduce the operator

$$\Delta = p\, |\psi\rangle\langle\psi| - q\, |\phi\rangle\langle\phi| \quad (4.27)$$

for diagonalizing it into the positive and negative subspaces, such that

$$\Delta = \lambda_+ |+\rangle\langle+| + \lambda_- |-\rangle\langle-| \quad (4.28)$$

where $\langle + | - \rangle = 0$, $\lambda_+ > 0$ and $\lambda_- < 0$. The optimal discrimination protocol provides for the measurement of $p\rho + q\sigma$ in the $\{|+\rangle, |-\rangle\}$ basis, then assigns to the positive subspace the state being ρ, the result is σ otherwise.

In order to understand when the protocol is realizable by means of LOCC, we make use of the criterion in eq. (4.5) for the vectors $|+\rangle$ and $|-\rangle$, namely we look for

$$\langle +_E | -_E \rangle = \langle +_O | -_O \rangle = 0. \quad (4.29)$$

The condition can be correctly inspected once we have chosen another basis $\{|\psi\rangle, |\psi^\perp\rangle\}$, by selecting the normalized vector $|\psi^\perp\rangle$ such that $\langle \psi | \psi^\perp \rangle = 0$ and

$$|\phi\rangle = \alpha\, |\psi\rangle + \beta\, |\psi^\perp\rangle \quad \text{for} \quad \beta = \sqrt{1 - \alpha^2} \in \mathbb{R}. \quad (4.30)$$

The operator Δ is hermitian and can be diagonalized through unitary matrices, which we express in the new basis as

$$U = \begin{pmatrix} U_{00} & U_{01} \\ U_{10} & U_{11} \end{pmatrix} \quad (4.31)$$

such that

$$|+\rangle = U\, |\psi\rangle = U_{00}\, |\psi\rangle + U_{01}\, |\psi^\perp\rangle, \quad (4.32)$$
$$|-\rangle = U\, |\psi^\perp\rangle = U_{10}\, |\psi\rangle + U_{11}\, |\psi^\perp\rangle, \quad (4.33)$$

[3] We are actually given the overlap amplitude $|\alpha|^2 = \mathrm{Tr}\left[|\psi\rangle\langle\psi| \cdot |\phi\rangle\langle\phi|\right] = |\langle\psi|\phi\rangle|^2$, which we assume to be real.

and

$$\Delta = U\left(\lambda_+ |\psi\rangle\!\langle\psi| + \lambda_- \left|\psi^\perp\right\rangle\!\left\langle\psi^\perp\right|\right) U^\dagger. \tag{4.34}$$

Remark. The matrix U has a zero entry if and only if U is either the identity or the swap operator[4], while neglecting the phase shifts. However, in both cases the matrix Δ is already diagonal and the two vectors $|\psi\rangle$, $|\phi\rangle$ are orthogonal, contrary to our hypothesis. We therefore assume the terms U_{ij} being non-zero from now on.

We introduce the amplitudes for the vectors $|+\rangle$, $|-\rangle$:

$$\begin{aligned}|+\rangle &= a_E |+_E\rangle + a_O |+_O\rangle \\ &= a_E\left(U_{00}\frac{\psi_E}{a_E}|\psi_E\rangle + U_{01}\frac{\psi_E^\perp}{a_E}\left|\psi_E^\perp\right\rangle\right) + a_O\left(U_{00}\frac{\psi_O}{a_O}|\psi_O\rangle + U_{01}\frac{\psi_O^\perp}{a_O}\left|\psi_O^\perp\right\rangle\right),\end{aligned} \tag{4.35}$$

$$\begin{aligned}|-\rangle &= b_E |-_E\rangle + b_O |-_O\rangle \\ &= b_E\left(U_{10}\frac{\psi_E}{b_E}|\psi_E\rangle + U_{11}\frac{\psi_E^\perp}{b_E}\left|\psi_E^\perp\right\rangle\right) + b_O\left(U_{10}\frac{\psi_O}{b_O}|\psi_O\rangle + U_{11}\frac{\psi_O^\perp}{b_O}\left|\psi_O^\perp\right\rangle\right),\end{aligned} \tag{4.36}$$

so that, from eqs. (4.32), (4.33), (4.35) and (4.36), we have

$$|a_E|^2 = |U_{00}|^2|\psi_E|^2 + |U_{01}|^2\left|\psi_E^\perp\right|^2, \qquad |a_O|^2 = |U_{00}|^2|\psi_O|^2 + |U_{01}|^2\left|\psi_O^\perp\right|^2, \tag{4.37}$$

$$|b_E|^2 = |U_{10}|^2|\psi_E|^2 + |U_{11}|^2\left|\psi_E^\perp\right|^2, \qquad |b_O|^2 = |U_{10}|^2|\psi_O|^2 + |U_{11}|^2\left|\psi_O^\perp\right|^2. \tag{4.38}$$

Taken for granted the above remark, we know that the amplitudes a_E and b_E are zero if and only if $\psi_E = \psi_E^\perp = 0$. Up to now, the value of $\psi_E^\perp$ still has no physical meaning, as it is not related to any property of the vectors $|\psi\rangle$, $|\phi\rangle$. Thus, we define the two quantities γ and δ such that

$$\langle\psi|\phi\rangle = \underbrace{\bar{\psi}_E\phi_E\langle\psi_E|\phi_E\rangle}_{\gamma} + \underbrace{\bar{\psi}_O\phi_O\langle\psi_O|\phi_O\rangle}_{\delta} = \gamma + \delta = \alpha \tag{4.39}$$

and rewrite the vector $\left|\psi^\perp\right\rangle$ as

$$\begin{aligned}\left|\psi^\perp\right\rangle &= \frac{1}{\beta}|\phi\rangle - \frac{\alpha}{\beta}|\psi\rangle = \psi_E^\perp\left|\psi_E^\perp\right\rangle + \psi_O^\perp\left|\psi_O^\perp\right\rangle \\ &= \frac{\psi_E^\perp}{\beta}\left(\frac{\phi_E}{\psi_E^\perp}|\phi_E\rangle - \frac{\alpha\psi_E}{\psi_E^\perp}|\psi_E\rangle\right) + \frac{\psi_O^\perp}{\beta}\left(\frac{\phi_O}{\psi_O^\perp}|\phi_O\rangle - \frac{\alpha\psi_O}{\psi_O^\perp}|\psi_O\rangle\right).\end{aligned} \tag{4.40}$$

The modulus squared of the $\left|\psi^\perp\right\rangle$ amplitudes can be promptly written down from eqs. (4.39) and (4.40):

$$\left|\psi_E^\perp\right|^2 = \frac{1}{|\beta|^2}\left(|\phi_E|^2 + |\alpha|^2|\psi_E|^2 - 2\,\mathrm{Re}\,\alpha\bar{\gamma}\right), \tag{4.41}$$

$$\left|\psi_O^\perp\right|^2 = \frac{1}{|\beta|^2}\left(|\phi_O|^2 + |\alpha|^2|\psi_O|^2 - 2\,\mathrm{Re}\,\alpha\bar{\delta}\right). \tag{4.42}$$

[4] The operator $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ swaps the basis elements.

Equation (4.41) tells us, that the hypothesis $\psi_E = \psi_E^\perp = 0$ is valid if and only if $\psi_E = \phi_E = 0$, leading to two disjoint cases. Either both vectors $|\psi\rangle$, $|\phi\rangle$ have no even parts or the measure has to range over both even and odd subspaces, in order to optimally discriminate the states, and belongs to the most general setting. In the former situation the amplitudes a_E and b_E are zero and the quest reduces to the quantum one, because the measurement between $|+\rangle$ and $|-\rangle$ is LOCC-implementable thanks to the protocol of Walgate et al. The statement is valid for the odd amplitudes correspondingly. The latter one, instead, constitutes the case of greatest interest as it introduces a distinctive trait of the Fermionic theory with respect to the quantum one: not all state pairs are locally *and* optimally discriminable.

At this stage, we derive and prove a necessary and sufficient condition for two pure states to be optimally discriminable through LOCC

Theorem 8. *Let $\rho = p\,|\psi\rangle\langle\psi|$ and $\sigma = q\,|\phi\rangle\langle\phi|$ be two pure and probabilistic states for $p, q > 0$ and $p + q = 1$. The they are optimally discriminable if and only if they satisfy*

$$[\Delta, P_E] = 0, \tag{4.43}$$

where

$$\Delta = p\,|\psi\rangle\langle\psi| - q\,|\phi\rangle\langle\phi| \tag{4.27}$$

and P_E is the projector onto the subspace $\mathcal{F}_0^n \otimes \mathcal{F}_0^m$.

Proof. We follow the optimal discrimination strategy proposed by Helstrom and introduce the operator Δ as in eq. (4.27) along with the eigenvectors $\{|+\rangle, |-\rangle\}$. We have already proved above that, if either $\psi_E = \phi_E = 0$ or $\psi_O = \phi_O = 0$, the states are discriminable through LOCC thanks to theorem 1 applied on the non-zero part. Indeed, states of the form

$$\rho = p\,|\psi_E\rangle\langle\psi_E| \quad \text{and} \quad \sigma = q\,|\phi_E\rangle\langle\phi_E| \tag{4.44}$$

or

$$\rho = p\,|\psi_O\rangle\langle\psi_O| \quad \text{and} \quad \sigma = q\,|\phi_O\rangle\langle\phi_O| \tag{4.45}$$

do satisfy eq. (4.43).

On the other hand, if none of the two conditions mentioned above are satisfied, the amplitudes a_E, a_O and b_E, b_O of vectors $|+\rangle$, $|-\rangle$, respectively, are non-zero. We can then show that eq. (4.29) is equivalent to

$$\langle +|P_E|-\rangle = \langle +|P_O|-\rangle = 0, \tag{4.46}$$

where P_O is the projector onto $\mathcal{F}_1^n \otimes \mathcal{F}_1^m$, since we assumed

$$\langle +|P_E|-\rangle = \bar{a}_E b_E \langle +_E|-_E\rangle\,, \tag{4.47}$$

$$\langle +|P_O|-\rangle = \bar{a}_O b_O \langle +_O|-_O\rangle\,. \tag{4.48}$$

If we evaluate the sum and difference of eq. (4.46), we come to

$$\langle +|P_E + P_O|-\rangle = 0 \tag{4.49}$$

$$\langle +|P_E - P_O|-\rangle = 0, \tag{4.50}$$

where the former equation is granted by the relations $P_E + P_O = I_0$, i. e. their sum is the projector onto $\mathcal{F}_0^{n+m}$, and $\langle +|-\rangle = 0$. We successfully reduced the LOCC condition to a single expression and can now infer a relevant propriety of the operator Δ. Indeed, we define the Hermitian operator

$$D = P_E - P_O, \tag{4.51}$$

and note that eq. (4.50) leads the restriction of operator D onto the space $\mathrm{Span}\{|\psi\rangle, |\phi\rangle\}$ to be diagonal in the basis $\{|+\rangle, |-\rangle\}$. Two operators are simultaneously diagonalizable if and only if they commute, i. e.

$$[\Delta, D] = 0 \tag{4.52}$$

or, given eq. (4.27) and that $P_O = I_0 - P_E$,

$$[p\,|\psi\rangle\langle\psi| - q\,|\phi\rangle\langle\phi|, P_E] = 0. \tag{4.53}$$

Hence, eq. (4.43) is valid if and only if eq. (4.29) is satisfied. We attain the optimal discrimination of the two states ρ and σ by measuring in the basis $\{|+\rangle, |-\rangle\}$ through the LOCC protocol of § 4.1. □

The above result of theorem 8 provides us with an expression to assess whether optimal discrimination is achievable through LOCC or not. Moreover, eq. (4.43) is basis independent and does not require us to first diagonalize the operator Δ, as eq. (4.29) instead does. The striking consequence of theorem 8 is, though, that we are not able to optimally distinguish any two pure states by only means of LOCC, contrary to QT. Lastly, we may overcome such limitation of those preparations not fulfilling eq. (4.43) by taking advantage of a shared entangled resource, as described in the protocol of § 4.2.

Chapter 5

Conclusion

In the first chapters of the thesis, we reviewed the modern literature on different subjects regarding quantum theory and the Fermionic quantum theory to collect sufficiently sharp tools for our development. Chapters 2 and 3 deal with state discrimination in QT, quantum LOCC protocols and the description of FQT to collect some already well-known results we recapitulate hereafter. Firstly, the very pragmatic notion of local operations and classical communication is of easy understanding. Nevertheless, it has a complex and contrived mathematical definition and still leaves some open problems, cf. [Chi+14]. Secondly, discrimination strategies relying on quantum SEP effects offers no real advantage over those based onto the subset of LOCC when we both consider any pair of orthogonal pure states, for the perfect discrimination problem, and non-orthogonal pure preparations, for the optimal one. Lastly, the FQT presents the same state set of the superselected qubit theory, though the notion of locality for the Fermionic transformations is subtly different, leading to an entanglement of distinct nature.

We managed to derive a condition for two Fermionic, pure and orthogonal states for being perfectly discriminable through LOCC. Thus, we proved that some Fermionic pure preparations are not perfectly distinguishable by only means of LOCC, contrary to the QT. The result of § 4.1 surprisingly does not imply that the SEP and LOCC sets have different discriminations capabilities. Quite the opposite, they are still equivalent: either they both discriminate pure orthogonal states or they do not. We still do not know if the above property belongs to the QT and FQT only, or it is related to some specific postulates the theories share with others.

The limitation of the theory can be overcome by taking advantage of a shared entangled ancilla. Namely, if Alice and Bob each owns a part of a system prepared in

$$|\omega\rangle = \frac{1}{\sqrt{2}}(|00\rangle + e^{i\varphi}|11\rangle),$$

they can achieve perfect discrimination through LOCC on the composite system, as shown in § 4.2. However, the remarkable result is that only a maximally entangled preparation enables LOCC ideal discrimination. If we plotted the discrimination efficiency of the entanglement assisted protocol as a function of the entanglement amount—consider the variable $|a|^2$ for instance—we would then have one for a completely entangled ancilla, zero otherwise. The reasons for such an extreme discontinuity are unknown to us and shall be better investigated in the future.

The perfect discriminability is a key feature of a theory for the optimal distinguishability of two pure non-orthogonal states. Indeed, we showed in § 4.3 that the optimal discrimination strategy of Helstrom for Fermionic preparations is given by the measurement onto two orthogonal vectors $\{|+\rangle, |-\rangle\}$, which diagonalize the particular operator Δ evaluated from the two considered states. We proved in this case too that not all Fermionic pure states are optimally discriminable by only means of LOCC. Furthermore, we provided an expression for assessing whether any two states are optimally LOCC-distinguishable without having to first diagonalize the operator Δ.

Further continuations of this work are the investigation of the feasibility through LOCC of unambiguous and inconclusive discrimination, as well as the study of the much harder quest of mixed state discrimination within the Fermionic theory.

Appendix A

Equidiagonalization of complex matrices

We prove hereafter that any $2^n \times 2^n$ complex matrix is equidiagonalizable by means of unitary matrices, that is we attain a new matrix whose diagonal elements are all equal. We start by proving the protocol for bidimensional matrices.

Theorem 9 (Equidiagonalization of $\mathsf{M}(2)$)**.** *Let M be a 2×2 complex matrix*

$$M = \begin{pmatrix} x & y \\ z & t \end{pmatrix}. \tag{A.1}$$

The matrix M is equidiagonalizable, i. e. there exists a unitary matrix $U \in \mathsf{U}(2)$ such that the diagonal elements of $UMU^\dagger$ are equal.

Proof. We consider the unitary matrix

$$U = \begin{pmatrix} \cos\theta & \sin\theta e^{i\omega} \\ \sin\theta e^{-i\omega} & -\cos\theta \end{pmatrix} \quad \text{where} \quad \theta, \omega \in [0, 2\pi], \ \det[U] = -1 \tag{A.2}$$

and look for ${UMU^\dagger}_{00} = {UMU^\dagger}_{11}$. We have

$$\begin{aligned} &\sin\theta\cos\theta\left(ze^{i\omega} + ye^{-i\omega}\right) + t\sin^2(\theta) + x\cos^2(\theta) \\ &\qquad = -\sin\theta\cos\theta\left(ze^{i\omega} + ye^{-i\omega}\right) + t\cos^2(\theta) + x\sin^2(\theta), \end{aligned} \tag{A.3}$$

hence

$$\sin(2\theta)\left(ze^{i\omega} + ye^{-i\omega}\right) + \cos(2\theta)(x - t) = 0 \tag{A.4}$$

as well as, for $ze^{i\omega} + ye^{-i\omega} \neq 0$,

$$\tan(2\theta)\left(ze^{i\omega} + ye^{-i\omega}\right) + (x - t) = 0. \tag{A.5}$$

Since we assume the matrix M not being equidiagonal yet, we take for granted that $x - t \neq 0$ and $\theta \neq k\frac{\pi}{2}$ for $k \in \mathbb{Z}$. The imaginary part of eq. (A.5) reads

$$\operatorname{Im} \frac{ze^{i\omega} + ye^{-i\omega}}{t - x} = 0, \tag{A.6}$$

and lets us determine the parameter

$$\omega = \arctan \frac{\mathrm{Im}\{x-t\}\,\mathrm{Re}\{z+y\} - \mathrm{Re}\{x-t\}\,\mathrm{Im}\{z+y\}}{\mathrm{Re}\{x-t\}\,\mathrm{Re}\{z-y\} + \mathrm{Im}\{x-t\}\,\mathrm{Im}\{z-y\}}. \tag{A.7}$$

Finally, we put eq. (A.7) into (A.4) and solve for θ

$$\tan(2\theta) = \frac{t-x}{ze^{i\omega} + ye^{-i\omega}} \in \mathbb{R} \tag{A.8}$$

or

$$\theta = \frac{\pi}{4} + \frac{\pi}{2}n, \quad n \in \mathbb{Z} \quad \text{for} \quad ze^{i\omega} + ye^{-i\omega} = 0. \tag{A.9}$$

Both eqs. (A.6) and (A.8) are solvable and lead us to the right choice of U for equidiagonalizing the matrix M. □

Next, we consider the particular set of complex matrices acting on spaces of dimension equal to 2^k as, e. g., those representing k qubits.

Theorem 10 (Equidiagonalization of $\mathsf{M}(2^k)$)**.** *Let M be a $2^k \times 2^k$ matrix for $k \in \mathbb{N}$, M is equidiagonalizable through unitary matrices.*

Proof. Thanks to theorem 9, we know how to write down the matrix $U(\theta, \omega)$ in order to get equal diagonal elements for $UMU^\dagger$ when $k = 2$. We use the same result to equidiagonalize several pairs of diagonal elements in M simultaneously.

From eqs. (A.3) and (A.8) we write the diagonal terms of $UMU^\dagger$ as

$$\begin{aligned} {UMU^\dagger}_{00} &= \sin\theta\cos\theta\left(ze^{i\omega} + ye^{-i\omega}\right) + t\sin^2(\theta) + x\cos^2(\theta) \\ &= (t-x)\frac{\sin\theta\cos\theta}{\tan(2\theta)} + (t-x)\sin^2(\theta) + x \\ &= (t-x)\left(\frac{\sin\theta\cos\theta}{\tan(2\theta)} + \sin^2(\theta)\right) + x \\ &= \frac{t+x}{2} = {UMU^\dagger}_{11} \end{aligned} \tag{A.10}$$

and learn that they are functions of the initial matrix diagonal-only elements. In case of $ze^{i\omega} + ye^{-i\omega} = 0$, the result is as in eq. (A.9) and leads to the same conclusion. Hence, let $M \in \mathsf{M}(2^k)$ and U' a unitary matrix, we consider the direct sum

$$U' = \bigoplus_{i=1}^{2^k} U_i(\theta, \omega) \tag{A.11}$$

of $U(\theta, \omega)$ matrices from theorem 9 and pairwise equidiagonalize k couples of diagonal terms of M. We then accordingly shuffle the basis terms such that we come to new pairs of the form

$$M' = U'MU'^\dagger = \begin{pmatrix} \ddots & & \\ & \boxed{\begin{matrix} \xi & \upsilon \\ \zeta & \tau \end{matrix}} & \\ & & \boxed{\begin{matrix} \xi & \gamma \\ \delta & \tau \end{matrix}} \\ & & & \ddots \end{pmatrix}. \tag{A.12}$$

At this point, we separately equidiagonalize the two pairs having diagonal terms ξ, τ and, as we shown in eq. (A.12), the resulting quartet has the diagonal elements equal to $\frac{\zeta+\xi}{2}$. We proved the equidiagonalization of $\mathsf{M}(4)$. Finally, larger matrices are equidiagonalized by simultaneously applying four times the protocol of theorem 9 on all quartets for achieving eight equal diagonal terms, eight times for equidiagonalizing the sixteen-elements group and so forth until all diagonal terms are equal. The algorithm requires $k2^{k-1}$ elementary operations. □

The proof for matrices of any dimensions is given in theorem 1 by enlarging the quantum system, i. e. by embedding the matrix in a larger space.

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